

DUALITY THEOREMS OF ÉTALE GERBES ON ORBIFOLDS

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ABSTRACT. Let G be a finite group and \mathcal{Y} a G -gerbe over an orbifold \mathcal{B} . A disconnected orbifold $\widehat{\mathcal{Y}}$ and a flat $U(1)$ -gerbe c on $\widehat{\mathcal{Y}}$ is canonically constructed from \mathcal{Y} . Motivated by a proposal in physics, we study a mathematical duality between the geometry of the G -gerbe \mathcal{Y} and the geometry of $\widehat{\mathcal{Y}}$ twisted by c . We prove several results and verify this duality in the contexts of non-commutative geometry and symplectic topology. In particular, we prove that the category of sheaves on \mathcal{Y} is equivalent to the category of c -twisted sheaves on $\widehat{\mathcal{Y}}$. When \mathcal{Y} is symplectic, we show, by a combination of techniques from non-commutative geometry and symplectic topology, that the Chen-Ruan orbifold cohomology of \mathcal{Y} is isomorphic to the c -twisted orbifold cohomology of $\widehat{\mathcal{Y}}$ as graded algebras.

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Date: May 24, 2010.

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1. INTRODUCTION

1.1. **Background.** The notion of an orbifold¹ was first introduced in [62] under the name “ V -manifold,” and was introduced in algebraic geometry in [29], now called a Deligne-Mumford stack. The term “orbifold” was coined by Thurston [65] during his study of 3-dimensional manifolds. Orbifolds are geometric objects that are locally modeled on quotients of manifolds by actions of finite groups. Introductory accounts about orbifolds can be found in [50], [5], and [42].

Besides being interesting in its own right, the theory of orbifolds can be applied in numerous areas, such as the study of moduli problems and quotient singularities. Moreover, there has been an increase of activity in the study of the stringy geometry of orbifolds. See [60], [61], and [5] for expository accounts.

In this paper, we study a special kind of orbifold called a *gerbe*. Let G be a finite group and $BG = [pt/G]$ denote the classifying orbifold of G . Roughly speaking, one can think of a G -gerbe over an orbifold \mathcal{B} as a principal BG -bundle over \mathcal{B} . Then in order to define a G -gerbe \mathcal{Y} over \mathcal{B} , one starts with an open cover $\{U_i\}$ of \mathcal{B} and specifies the following data:

$$(1.1) \quad \begin{aligned} \varphi_{ij} &\in \text{Aut}(G) \quad \text{for each double overlap } U_{ij} := U_i \cap U_j, \text{ and} \\ g_{ijk} &\in G \quad \text{for each triple overlap } U_{ijk} := U_i \cap U_j \cap U_k, \end{aligned}$$

so that the following constraints are satisfied:

$$(1.2) \quad \begin{aligned} \varphi_{jk} \circ \varphi_{ij} &= \text{Ad}_{g_{ijk}} \circ \varphi_{ik}, \quad \text{on } U_{ijk}, \\ g_{jkl}g_{ijl} &= \varphi_{kl}(g_{ijk})g_{ikl}, \quad \text{on } U_i \cap U_j \cap U_k \cap U_l. \end{aligned}$$

Here, $\text{Ad}_g : G \rightarrow G$ denotes the map of conjugation by g . The data in (1.1) is then used to glue BG together to form a G -gerbe \mathcal{Y} together with an associated map $\mathcal{Y} \rightarrow \mathcal{B}$.

We easily see that BG is the unique G -gerbe over a point. Gerbes arise naturally from ineffective group actions. For example, let M be a manifold and H be a compact group that acts on M with finite stabilizers at every point. The quotient space $[M/H]$ is an orbifold. Suppose that G is a finite normal subgroup of H such that the induced action of G on M is trivial. Then there is an induced action of the quotient group $Q := H/G$ on M . The orbifold $[M/H]$ defines a G -gerbe

$$[M/H] \rightarrow [M/Q]$$

over the orbifold $[M/Q]$. In general, gerbes play an important role in the structure theory of orbifolds. For example, given an orbifold \mathcal{X} there is a finite group G and a *reduced* orbifold \mathcal{X}' such that \mathcal{X} is a

¹Throughout this paper we consider orbifolds which are not necessarily reduced.

G -gerbe over \mathcal{X}' . See [12, Proposition 4.6]. Introductory accounts about gerbes can be found in [33], [30], and [44].

Gerbes, in a more general context, are useful tools for approaching various problems that are seemingly unrelated to orbifolds. They can be applied to the theory of non-abelian cohomology [33], loop spaces and characteristic classes [18], the Dixmier-Douady class, continuous trace C^* -algebras, index theory ([57] and [21]), the comparison between the Brauer group and the cohomological Brauer group of a scheme [28], and the period-index problem [46]. Furthermore, in physics, gerbes are intimately connected with the study of discrete torsion. See, e.g., [69] and [63].

The purpose of this paper is to study the geometry and topology of G -gerbes. Our study is motivated and inspired by results in the physics paper [36]. Given a G -gerbe $\mathcal{Y} \rightarrow \mathcal{B}$, [36] gives the construction of a disconnected space $\widehat{\mathcal{Y}}$ with a map $\widehat{\mathcal{Y}} \rightarrow \mathcal{B}$ and a flat $U(1)$ -gerbe c on $\widehat{\mathcal{Y}}$. This construction is reviewed in Section 1.2 below. The main point of [36] is the conjecture which asserts that the conformal field theories on the G -gerbe \mathcal{Y} are equivalent to the corresponding conformal field theories on $\widehat{\mathcal{Y}}$ twisted by the B -field c . This conjecture is not mathematically well-defined for a number of reasons. For instance, the notion of conformal field theory is not yet mathematically well-defined. However, it suggests the existence of a certain duality between the G -gerbe \mathcal{Y} and the pair $(\widehat{\mathcal{Y}}, c)$. Our viewpoint toward this conjecture is that it suggests the following claim:

(*) *The geometry/topology of the G -gerbe \mathcal{Y} is equivalent to the geometry/topology of $\widehat{\mathcal{Y}}$ twisted by c .*

The claim (*) reveals a deep and highly nontrivial connection between different geometric spaces. Let us look at the simplest example of a G -gerbe, namely, a G -gerbe over a point, (i.e., $\mathcal{B} = pt$ and $\mathcal{Y} = [pt/G] = BG$). Then the dual orbifold $\widehat{\mathcal{Y}}$ is the discrete set \widehat{G} , the space of isomorphism classes of irreducible G -representations, and the $U(1)$ -gerbe c on $\widehat{\mathcal{Y}}$ is trivial. In this case, the claim (*) states that the geometry/topology of the classifying space BG is equivalent to the geometry/topology of the discrete set \widehat{G} . However, such a relationship is not clear at all at the level of spaces. For example, when $G = \mathbb{Z}_2$, there does not seem to be any obvious geometric connection between the space $B\mathbb{Z}_2$ (interpreted either as an orbifold $[pt/\mathbb{Z}_2]$ or as the space \mathbb{RP}^∞) and the space $\widehat{\mathbb{Z}_2}$. In general, to our best knowledge there is no known geometric relation at the level of spaces between a G -gerbe \mathcal{Y} and the orbifold $\widehat{\mathcal{Y}}$ with the $U(1)$ -gerbe c .

One observes that a natural place where both BG and \widehat{G} appear is representation theory, since BG encodes information about principal G -bundles, and \widehat{G} is defined to be the set of isomorphism classes of irreducible G -representations. Noncommutative geometry is a powerful modern approach to representation theory. Thus, it makes sense to consider possible relations between BG and \widehat{G} in noncommutative geometry. In noncommutative geometry, BG is represented by the group algebra $\mathbb{C}G$, and \widehat{G} is represented by the commutative algebra, $C(\widehat{G})$, of functions on \widehat{G} . It is a classical result that the group algebra $\mathbb{C}G$ is Morita equivalent to the algebra $C(\widehat{G})$. We can interpret this as saying that the two spaces BG and \widehat{G} are “equivalent” from the viewpoint of noncommutative geometry. This observation strongly suggests that noncommutative geometry naturally relates the two geometries of \mathcal{Y} and $(\widehat{\mathcal{Y}}, c)$, which appear to be very different in the classical geometric/topological viewpoints.

Indeed, the main theme of this paper is using tools from noncommutative geometry to build a bridge connecting the G -gerbe \mathcal{Y} and the dual $\widehat{\mathcal{Y}}$ with the $U(1)$ -gerbe c . Such a bridge turns out to be very useful. We will prove a number of results that show (*) is true in the contexts of both noncommutative geometry and symplectic topology, and in particular, in Gromov-Witten theory. These results are discussed in the rest of the introduction.

1.2. The dual of an étale gerbe. Let G be a finite group and \mathcal{B} be an orbifold. Given a G -gerbe over \mathcal{B} ,

$$\mathcal{Y} \rightarrow \mathcal{B},$$

we describe the dual of the gerbe \mathcal{Y} according to [36]. Consider the group of outer automorphisms of G ,

$$Out(G) = Aut(G)/Inn(G),$$

i.e., the quotient of the group $Aut(G)$ of automorphisms of G and the normal subgroup $Inn(G)$ of inner automorphisms of G . Given a G -gerbe $\mathcal{Y} \rightarrow \mathcal{B}$, there is a naturally defined $Out(G)$ -bundle,

$$\overline{\mathcal{Y}} \rightarrow \mathcal{B},$$

called the *band* of this gerbe. In the description (1.1) of the G -gerbe \mathcal{Y} , let $\phi_{ij} \in Out(G)$ be the image of φ_{ij} under the quotient map $Aut(G) \rightarrow Out(G)$. By (1.2) we have $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on U_{ijk} . Thus, the collection $\{\phi_{ij}\}$ defines an $Out(G)$ -bundle $\overline{\mathcal{Y}}$ over \mathcal{B} .

Let \widehat{G} denote the (finite) set of isomorphism classes of irreducible representations of G . The cardinality of \widehat{G} is equal to the number of conjugacy classes of G . We also view \widehat{G} as a disjoint union of points. A right action of $Out(G)$ on \widehat{G} is defined as follows. Given an irreducible G -representation $\rho : G \rightarrow End(V_\rho)$ and $\phi \in Aut(G)$, the composite map

$$\rho \circ \phi : G \rightarrow End(V_\rho)$$

is an irreducible representation of G . The action of ϕ on the class $[\rho]$ is defined to be $[\rho \circ \phi]$. This defines a right action of $Out(G)$ on \widehat{G} because inner automorphisms preserve isomorphism classes of irreducible representations of G . Note that the isomorphism class of the 1-dimensional trivial representation of G is fixed by this $Out(G)$ action.

Following [36], we define the dual space to be the associated bundle

$$(1.3) \quad \widehat{\mathcal{Y}} := [(\overline{\mathcal{Y}} \times \widehat{G})/Out(G)].$$

There is a natural map, $\widehat{\mathcal{Y}} \rightarrow \mathcal{B}$, induced from the map $\overline{\mathcal{Y}} \rightarrow \mathcal{B}$. It is easy to see that $\widehat{\mathcal{Y}}$ decomposes into a union of $Out(G)$ orbits, and $\widehat{\mathcal{Y}}$ decomposes into a union of components with respect to the $Out(G)$ orbits.

Remark 1.1. We say that the G -gerbe $\mathcal{Y} \rightarrow \mathcal{B}$ has a trivial band if the $Out(G)$ -bundle $\overline{\mathcal{Y}} \rightarrow \mathcal{B}$ is endowed with a section (hence is trivialized by this section). In this case, the dual, $\widehat{\mathcal{Y}}$, is a disjoint union of several copies of \mathcal{B} , and the map $\widehat{\mathcal{Y}} \rightarrow \mathcal{B}$ restricts to the identity on each copy.

Next, we define a $U(1)$ -gerbe, c , on $\widehat{\mathcal{Y}}$. For each isomorphism class $[\rho] \in \widehat{G}$, we fix a representation $\rho : G \rightarrow End(V_\rho)$. To each point $(x, [\rho]) \in \widehat{\mathcal{Y}}$, we assign the vector space V_ρ . This defines a family of vector spaces over $\widehat{\mathcal{Y}}$, which is in general *not* a vector bundle over $\widehat{\mathcal{Y}}$. The obstruction to forming a vector bundle over $\widehat{\mathcal{Y}}$ with the fiber over $(x, [\rho])$ being V_ρ is a $U(1)$ -gerbe, c^{-1} , on $\widehat{\mathcal{Y}}$. The $U(1)$ -gerbe c over $\widehat{\mathcal{Y}}$ is obtained from c^{-1} by applying the group homomorphism $U(1) \rightarrow U(1), a \mapsto a^{-1}$. As observed in [36], the $U(1)$ -gerbe c is flat, and the isomorphism class of c is a *torsion* class in the cohomology $H^2(\widehat{\mathcal{Y}}, U(1))$.

Another way to understand the $U(1)$ -gerbe c is the following. The failure for the V_ρ 's to form a vector bundle over $\widehat{\mathcal{Y}}$ is due to the fact that they glue *up to scalars*. In other words, the V_ρ 's glue to a *twisted sheaf* (in the sense of [20]), where the twist is given by a $U(1)$ -valued 2-cocycle on $\widehat{\mathcal{Y}}$. This twisted sheaf is equivalent (see [45]) to a sheaf on a $U(1)$ -gerbe, c^{-1} , over $\widehat{\mathcal{Y}}$.

Remark on terminology. It is well-known that the isomorphism classes of flat $U(1)$ -gerbes over a space \mathcal{X} are in bijective correspondence with the torsion classes of the cohomology group $H^2(\mathcal{X}, U(1))$. If we fix a sufficiently fine open cover, $\{V_i\}$, of \mathcal{X} , then the flat $U(1)$ -gerbes on \mathcal{X} (not their isomorphism classes) are in bijective correspondence with the $U(1)$ -valued Čech 2-cocycles with respect to the cover $\{V_i\}$, which represents the torsion classes in $H^2(\mathcal{X}, U(1))$.

In our study of the G -gerbe $\mathcal{Y} \rightarrow \mathcal{B}$ and the dual pair $(\widehat{\mathcal{Y}}, c)$, we often choose a presentation of the orbifold \mathcal{B} arising from a sufficiently fine open cover of \mathcal{B} . Such a presentation yields an open cover of $\widehat{\mathcal{Y}}$, and we often represent the $U(1)$ -gerbe c by a $U(1)$ -valued Čech 2-cocycle with respect to this cover. In view of the aforementioned correspondence, in what follows, we abuse the notation and let c denote either the $U(1)$ -gerbe on $\widehat{\mathcal{Y}}$ or the $U(1)$ -valued Čech 2-cocycle on $\widehat{\mathcal{Y}}$. We will call c a $U(1)$ -gerbe if a presentation of $\widehat{\mathcal{Y}}$ is not chosen and call c a $U(1)$ -valued 2-cocycle if a presentation of $\widehat{\mathcal{Y}}$ is chosen.

1.3. Gerbes arising from group extensions. The simplest examples of G -gerbes other than BG itself are of the form $BH \rightarrow BQ$, where H and Q are finite groups, and BH (respectively, BQ) is a classifying space of H (respectively, Q). Such an example arises from an extension of a finite group Q by G , namely, the exact sequence

$$1 \longrightarrow G \longrightarrow H \longrightarrow Q \longrightarrow 1.$$

Finite group extension is a well-studied classical subject in group theory, dating back to the time of Frobenius, Schur, and Clifford. Our study of the gerbe $BH \rightarrow BQ$ uses knowledge about finite group extensions extensively.

By construction, the dual space of BH is given by $\widehat{BH} = [\widehat{G}/Q]$. Here, the (right) Q -action on \widehat{G} is defined by a group homomorphism² $Q \rightarrow \text{Out}(G)$, which is constructed as follows. Choose³ a section, $s : Q \rightarrow H$, of the group extension such that $s(1) = 1$. Given $q \in Q$, we define an automorphism of G by

$$G \ni g \mapsto \text{Ad}_{s(q)}(g).$$

The homomorphism $Q \rightarrow \text{Out}(G)$ sends q to the image of the above automorphism of G . The $U(1)$ -gerbe on \widehat{BH} can be represented by a function $c : \widehat{G} \times Q \times Q \rightarrow U(1)$. See Proposition 3.1 for more details.

One object that is naturally associated with the group H is its group algebra $\mathbb{C}H$. Given the Q -action on \widehat{G} and the function c , one can define the *twisted groupoid algebra*

$$C(\widehat{G} \rtimes Q, c).$$

The construction, which is a special case of a construction in [68], is explained in Section 3.2.

Our first result for the gerbe $BH \rightarrow BQ$ is the following:

Theorem 1.2 (=Theorem 3.3). *The group algebra $\mathbb{C}H$ is Morita equivalent to the twisted groupoid algebra $C(\widehat{G} \rtimes Q, c)$.*

This theorem can be interpreted as *Mackey's machine* in the case of finite group extensions, which is well-studied. However, our formulation using the language of Morita equivalence seems to be new. The generalization of this theorem serves as a crucial step in our study of G -gerbes. We prove this theorem by explicit constructions of bimodules that realize the Morita equivalence. These constructions also allow us to analyze the structure of the induced isomorphism $I : Z(\mathbb{C}H) \rightarrow Z(C(\widehat{G} \rtimes Q, c))$ between centers. See Proposition 3.4.

²The group homomorphism $Q \rightarrow \text{Out}(G)$ defines an $\text{Out}(G)$ -bundle over BQ , which is the band of the G -gerbe $BH \rightarrow BQ$.

³It is easy to see that the resulting homomorphism, $Q \rightarrow \text{Out}(G)$, does not depend on the choice of such a section.

1.4. Non-commutative geometry. Theorem 1.2 is best understood in the context of non-commutative geometry. According to the view of non-commutative differential geometry à la A. Connes, the geometry of an orbifold is encoded in the Morita equivalence class of the groupoid algebras of its groupoid presentations⁴, and the geometry of an orbifold with a $U(1)$ -gerbe is represented by the Morita equivalence class of the associated twisted groupoid algebras. Therefore, Theorem 1.2 can be interpreted as saying that the non-commutative geometry of BH is equivalent to the non-commutative geometry of the dual (\widehat{BH}, c) . This provides an example of the claim (\star) .

One of the main results of this paper is a generalization of Theorem 1.2. Let \mathcal{B} be an orbifold, and $\mathcal{Y} \rightarrow \mathcal{B}$ be a G -gerbe over \mathcal{B} . As before, we denote by $\widehat{\mathcal{Y}}$, the dual space of the G -gerbe and by c , the $U(1)$ -gerbe on $\widehat{\mathcal{Y}}$. The Mackey machine provides a beautiful bridge between \mathcal{Y} and $(\widehat{\mathcal{Y}}, c)$ in the context of noncommutative geometry. As explained in Section 4.2, we pick groupoid presentations \mathfrak{H} (respectively, Ω) of \mathcal{Y} (respectively, \mathcal{B}) so that the dual space $\widehat{\mathcal{Y}}$ is represented by a transformation groupoid

$$\widehat{G} \rtimes \Omega.$$

The $U(1)$ -gerbe on $\widehat{\mathcal{Y}}$ can be presented by a locally constant groupoid 2-cocycle c . See Propositions 4.6-4.7. Let \mathcal{A} be a \mathfrak{H} -sheaf of unital algebras, and $\widetilde{\mathcal{A}}$ be the corresponding $\widehat{G} \rtimes \Omega$ -sheaf defined by \mathcal{A} . We consider the cross-product algebra $\mathcal{A} \rtimes \mathfrak{H}$ associated with the groupoid \mathfrak{H} and the *twisted* cross-product algebra $\widetilde{\mathcal{A}} \rtimes_c (\widehat{G} \rtimes \Omega)$ associated with $\widehat{G} \rtimes \Omega$ and c , as c is locally constant. We prove the following:

Theorem 1.3 (=Theorem 4.8). *The crossed product algebra $\mathcal{A} \rtimes \mathfrak{H}$ is Morita equivalent to the twisted crossed product algebra $\widetilde{\mathcal{A}} \rtimes_c (\widehat{G} \rtimes \Omega)$.*

When \mathcal{A} is the sheaf \mathcal{C}^∞ of smooth functions on \mathfrak{H}_0 , Theorem 1.3 shows that the groupoid algebra $\mathcal{C}_c^\infty(\mathfrak{H})$ is Morita equivalent to the c -twisted groupoid algebra $\mathcal{C}^\infty \rtimes_c (\widehat{G} \rtimes \Omega)$.

Consider the symplectic case, namely, the base \mathcal{B} is assumed to be symplectic. Then both the gerbe \mathcal{Y} and its dual $\widehat{\mathcal{Y}}$ can be equipped with symplectic structures pulled back from the one on \mathcal{B} . In this case, \mathfrak{H}_0 is equipped with a \mathfrak{H} -invariant symplectic form, and a \mathfrak{H} -sheaf $\mathcal{A}^{((h))}$ of deformation quantization on \mathfrak{H}_0 is constructed in [64] via Fedosov's construction. The crossed product algebra $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$ is a deformation quantization of the groupoid algebra $\mathcal{C}_c^\infty(\mathfrak{H})$, and $\widetilde{\mathcal{A}}^{((h))} \rtimes_c (\widehat{G} \rtimes \Omega)$ is a deformation quantization of the c -twisted groupoid algebra. Because our cocycle c is locally constant, the algebra $\widetilde{\mathcal{A}}^{((h))} \rtimes_c (\widehat{G} \rtimes \Omega)$ can be constructed by following the exact same method that is described in [64], which is recalled in Section 2.3. We refer the readers to [16] for a general discussion of deformation quantizations of gerbes. Theorem 1.3 shows that

Corollary 1.4. *The two deformation quantizations $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$ and $\widetilde{\mathcal{A}}^{((h))} \rtimes_c (\widehat{G} \rtimes \Omega)$ are Morita equivalent.*

We can interpret Theorem 1.3 as saying that the non-commutative differential geometry of the gerbe $\mathcal{Y} \rightarrow \mathcal{B}$ is equivalent to the non-commutative differential geometry of the dual pair $(\widehat{\mathcal{Y}}, c)$, proving the claim (\star) in full generality in the context of non-commutative geometry.

Categories of sheaves can be used to provide an algebro-geometric approach to non-commutative spaces. More precisely, in this approach, one studies geometric properties of spaces by considering properties of their categories of sheaves. For the G -gerbe \mathcal{Y} , we consider its sheaves. For the dual pair $(\widehat{\mathcal{Y}}, c)$, we consider the c -twisted sheaves over $\widehat{\mathcal{Y}}$. See [20], and [45] for detailed introductions to twisted sheaves.

⁴It is known that groupoid algebras arising from different groupoid presentations of the same orbifold are Morita equivalent.

Theorem 1.5 (=Theorem 7.11). *The abelian category of sheaves on \mathcal{Y} is equivalent to the abelian category of c -twisted sheaves on $\widehat{\mathcal{Y}}$.*

In the algebraic context, i.e., when both \mathcal{Y} and \mathcal{B} are Deligne-Mumford stacks, this theorem is also valid for categories of (quasi-)coherent sheaves. In this context, this theorem can also be interpreted as saying that the non-commutative algebraic geometry of the gerbe $\mathcal{Y} \rightarrow \mathcal{B}$ is equivalent to the non-commutative algebraic geometry of the dual $(\widehat{\mathcal{Y}}, c)$, proving the claim (\star) in full generality in the context of non-commutative algebraic geometry. This theorem suggests that in algebraic geometry a suitably defined theory of counting invariants of (semi)stable sheaves on \mathcal{Y} should be equivalent to an analogous theory of counting invariants of (semi)stable c -twisted sheaves on $\widehat{\mathcal{Y}}$. See [24] for the progress in this direction.

1.5. Hochschild cohomology. The Hochschild cohomology of an (associative) algebra is an important invariant of the algebra that depends only on the Morita equivalence class of the algebra. It also plays an important role in non-commutative geometry. See, for example, [26] and [47]. Motivated by this, we study the Hochschild cohomology of the algebras $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ and $\widetilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega)$ in Corollary 1.4.

In our joint work [56] with M. Pflaum and H. Posthuma, we found a beautiful connection between Hochschild cohomology and symplectic topology. Namely, we proved that the Hochschild cohomology of the deformation quantization of the groupoid algebra arising from a presentation of a symplectic orbifold \mathcal{X} is additively isomorphic to the shifted de Rham cohomology $H^{\bullet-\ell}(I\mathcal{X})((\hbar))$ (with coefficients in the field $\mathbb{C}((\hbar))$) of the *inertia orbifold*

$$I\mathcal{X} := \{(x, (g)) | x \in \mathcal{X}, (g) \subset \text{Iso}(x) \text{ conjugacy class of the isotropy subgroup } \text{Iso}(x)\}.$$

Here, the grading shift ℓ is given by the codimensions of the embeddings of components of $I\mathcal{X}$ into \mathcal{X} . Applying this result to the algebra $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ arising from the G -gerbe \mathcal{Y} , we obtain

$$H^{\bullet-\ell}(I\mathcal{Y})((\hbar)) \cong HH^{\bullet}(\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}, \mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}).$$

In this paper, we generalize the calculation of [56] to treat the Hochschild cohomology of the deformation quantization of a twisted groupoid algebra. This yields the following:

Theorem 1.6 (=Theorem 4.15). *The Hochschild cohomology of the algebra $\widetilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega)$ is equal to the c -twisted de Rham cohomology $H^{\bullet-\ell}(I\widehat{\mathcal{Y}}, c)((\hbar))$ of the orbifold $I\widehat{\mathcal{Y}}$ with the shifting defined by the codimensions, ℓ , of the embeddings of components of $I\widehat{\mathcal{Y}}$ into $\widehat{\mathcal{Y}}$,*

$$HH^{\bullet}(\widetilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega), \widetilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega)) \cong H^{\bullet-\ell}(I\widehat{\mathcal{Y}}, c)((\hbar)),$$

where $H^{\bullet}(I\widehat{\mathcal{Y}}, c)$ is the de Rham cohomology of $I\widehat{\mathcal{Y}}$ with coefficients in a line bundle \mathcal{L}_c , which is naturally defined by the $U(1)$ -gerbe c .

Since Morita equivalent algebras have isomorphic Hochschild cohomologies, Theorem 1.6 and Corollary 1.4 yield the following result:

Theorem 1.7 (=Theorem 4.16). *There are isomorphisms of cohomologies,*

$$\begin{aligned} H^{\bullet-\ell}(I\mathcal{Y})((\hbar)) &\cong HH^{\bullet}(\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}, \mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}) \\ &\cong HH^{\bullet}(\widetilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega), \widetilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega)) \cong H^{\bullet-\ell}(I\widehat{\mathcal{Y}}, c)((\hbar)). \end{aligned}$$

Moreover, the above isomorphisms yield an isomorphism of graded $\mathbb{C}((\hbar))$ -vector spaces,

$$H^{\bullet-\text{age}}(I\mathcal{Y})((\hbar)) \cong H^{\bullet-\text{age}}(I\widehat{\mathcal{Y}}, c)((\hbar)),$$

where the vector spaces are equipped with the age grading as defined in equation (2.1) (see also Definition 5.2).

In fact, this isomorphism is valid over \mathbb{C} , yielding an isomorphism

$$(1.4) \quad H^{\bullet-\text{age}}(I\mathcal{Y}, \mathbb{C}) \cong H^{\bullet-\text{age}}(I\widehat{\mathcal{Y}}, c, \mathbb{C})$$

of graded \mathbb{C} -vector spaces.

This theorem has important applications to the symplectic geometry/topology of G -gerbes, which we discuss next.

1.6. Chen-Ruan orbifold cohomology. Chen-Ruan orbifold cohomology, which was first introduced in [23], is a very important object in the theory of orbifolds, and has generated a lot of exciting research in recent years. A quick review of the construction of the Chen-Ruan orbifold cohomology is given in Section 5.1. For an almost complex orbifold \mathcal{X} , its Chen-Ruan cohomology, denoted by

$$H_{CR}^{\bullet}(\mathcal{X}, \mathbb{C}),$$

is additively the cohomology $H^{\bullet}(I\mathcal{X}, \mathbb{C})$ of the inertia orbifold, $I\mathcal{X}$, of \mathcal{X} , equipped with a shifted grading, a non-degenerate pairing called *orbifold Poincaré pairing*, and an associative product structure called the *Chen-Ruan orbifold cup product*. Structure constants of the Chen-Ruan orbifold cup product are defined to be certain integrals over the 2-multi-sector, $\mathcal{X}_{(2)}$, of \mathcal{X} involving the *obstruction bundle* $Ob_{\mathcal{X}}$. See Section 5.1 for more details.

Given an almost complex orbifold \mathcal{X} with a flat $U(1)$ -gerbe c , Y. Ruan [59] introduced the notion of the c -twisted orbifold cohomology of \mathcal{X} . This construction is also reviewed in Section 5.1. The c -twisted orbifold cohomology of \mathcal{X} , denoted by

$$H_{orb}^{\bullet}(\mathcal{X}, c, \mathbb{C}),$$

is additively the cohomology $H^{\bullet}(I\mathcal{X}, c, \mathbb{C}) := H^{\bullet}(I\mathcal{X}, \mathcal{L}_c)$ of $I\mathcal{X}$ with coefficients in the line bundle \mathcal{L}_c . The line bundle \mathcal{L}_c is naturally defined from the $U(1)$ -gerbe c and is an example of an *inner local system* [59]. The groups $H_{orb}^{\bullet}(\mathcal{X}, c, \mathbb{C})$ are equipped with a shifted grading, a non-degenerate pairing, and an associative product structure defined in ways similar to their counterparts in the Chen-Ruan orbifold cohomology. In particular, the structure constants of the product are also defined as certain integrals over the 2-multi-sector involving the obstruction bundle. See Section 5.1 for more details.

Consider a G -gerbe \mathcal{Y} over a compact symplectic orbifold \mathcal{B} and its dual pair $(\widehat{\mathcal{Y}}, c)$. Then \mathcal{Y} and $\widehat{\mathcal{Y}}$ are equipped with symplectic structures coming from the structure of \mathcal{B} . We equip \mathcal{Y} and $\widehat{\mathcal{Y}}$ with compatible almost complex structures and consider the two cohomology groups $H_{CR}^{\bullet}(\mathcal{Y}, \mathbb{C})$ and $H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C})$. By (1.4), there is an isomorphism of graded \mathbb{C} -vector spaces, i.e., $H_{CR}^{\bullet}(\mathcal{Y}, \mathbb{C}) \simeq H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C})$. We improve this in the following:

Theorem 1.8 (see Theorem 5.10). *There is an isomorphism*

$$(1.5) \quad H_{CR}^{\bullet}(\mathcal{Y}, \mathbb{C}) \simeq H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C})$$

of graded \mathbb{C} -algebras.

We view this theorem as the realization of (\star) at the level of Chen-Ruan orbifold cohomology rings. The isomorphism in this theorem is also compatible with (twisted) orbifold Poincaré pairings. See Corollary 5.9.

The proof of this theorem, given in Sections 5.2-5.3, amounts to showing that the additive isomorphism $H^{\bullet-\text{age}}(I\mathcal{Y}, \mathbb{C}) \cong H^{\bullet-\text{age}}(I\widehat{\mathcal{Y}}, c, \mathbb{C})$, obtained in Theorem 1.7, is in fact a ring isomorphism. The Morita equivalence bimodule in the proof of Theorem 1.3 gives an explicit formula of this isomorphism. To prove that the isomorphism in Theorem 1.7 preserves the ring structure, we need to compare the structure constants of the orbifold cup products. We first establish, in Proposition 5.7, a comparison result between the obstruction bundles $Ob_{\mathcal{Y}}$, $Ob_{\widehat{\mathcal{Y}}}$, and the obstruction bundle, $Ob_{\mathcal{B}}$, of the base \mathcal{B} . This reduces the question to comparing certain cohomology classes on the 2-multi-sector $\mathcal{B}_{(2)}$ of \mathcal{B} . See

(5.13). The proof of (5.13), given in Section 5.3, is achieved by carefully examining the isomorphism in Theorem 1.7 via representation theory of finite groups.

We should point out that Theorem 1.8 is somewhat surprising. Although, in some special examples (such as the toric case [11]), one can construct the isomorphism (1.5) directly, in general, it is not clear *at all* that the two cohomologies $H_{CR}^\bullet(\mathcal{Y}, \mathbb{C})$ and $H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ are equal even as vector spaces, since the G -gerbe \mathcal{Y} and the $U(1)$ -gerbe c on $\widehat{\mathcal{Y}}$ are related through representation theory and their geometric connections are somewhat obscure. Noncommutative geometry (and, in particular, Morita equivalence) provides us the right tool to extract the geometric information from representation theory. From this perspective, our construction of the isomorphism (1.5) via Morita equivalence and the connection between Hochschild cohomology and orbifold cohomology is very natural and, so far, the only known construction that works in general. Furthermore, the result that the two orbifold cohomologies are isomorphic as rings is not a formal consequence of our results on Hochschild cohomology. In [56], it is shown that the ring structure on $H_{CR}^\bullet(\mathcal{Y}, \mathbb{C})$ (and also $H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$) defined by the product on the Hochschild cohomology of the orbifold groupoid algebra is closely related, but *not* isomorphic, to the Chen-Ruan cup product. There are certain subtle but crucial differences between the Hochschild and Chen-Ruan cup products. Our proof of the fact that (1.5) is indeed a ring isomorphism with respect to Chen-Ruan cup products uses some delicate and important properties of the isomorphism (1.5). We view our proof of Theorem 1.8 as a successful application of non-commutative geometric techniques to the study of symplectic topology.

1.7. Gromov-Witten theory. Chen-Ruan orbifold cohomology and twisted orbifold cohomology can be considered as part of a bigger and richer theory called Gromov-Witten theory. Let \mathcal{X} be a compact symplectic orbifold. The Gromov-Witten theory of \mathcal{X} is the study of Gromov-Witten invariants of \mathcal{X} , which are integrals of certain naturally defined cohomology classes over moduli spaces of orbifold stable maps to \mathcal{X} . These invariants may be organized into a generating function, $\mathcal{D}_{\mathcal{X}}$, called the total descendant potential of \mathcal{X} , whose properties reflect the structures of Gromov-Witten invariants.

The Gromov-Witten theory of orbifolds is constructed in the work [22] in symplectic geometry and in the works [2, 3] in algebraic geometry. It has been a very active research area in the past few years. Expository accounts of this theory can also be found in [1] and [66].

Given a flat $U(1)$ -gerbe, c , on a compact symplectic orbifold \mathcal{X} , a “twist” of the Gromov-Witten theory of \mathcal{X} by c is constructed in the work [55]. The main ingredients here are the c -twisted Gromov-Witten invariants of \mathcal{X} . These invariants are integrals of certain naturally defined cohomology classes over the moduli spaces of orbifold stable maps to \mathcal{X} , and they can be organized into a generating function, $\mathcal{D}_{\mathcal{X}, c}$, called the c -twisted total descendant potential of \mathcal{X} .

It is known that for Calabi-Yau target spaces the Gromov-Witten theory may be understood as the mathematical version of a topological twist of a conformal field theory called non-linear sigma model. Since the original physics conjecture on the duality of étale gerbes concerns the equivalence of conformal field theories, it is very natural to consider the claim (\star) in the context of Gromov-Witten theory.

Let \mathcal{Y} be a G -gerbe over a compact symplectic orbifold \mathcal{B} and $(\widehat{\mathcal{Y}}, c)$ be its dual pair. The claim (\star) in the context of Gromov-Witten theory is naturally formulated in the following:

Conjecture 1.9. *There is an equality of generating functions*

$$\mathcal{D}_{\mathcal{Y}} = \mathcal{D}_{\widehat{\mathcal{Y}}, c},$$

after suitable changes of variables.

One can also formulate an analogue of Conjecture 1.9 for the *ancestor potentials* (see, e.g., [34, Section 5] for the definition of ancestor potential). In order for Conjecture 1.9 to possibly be true, it is necessary that the cohomology groups $H_{CR}^\bullet(\mathcal{Y}, \mathbb{C})$ and $H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ be isomorphic. Therefore, Theorem 1.8 provides the first step towards an approach to Conjecture 1.9 in general.

So far, progress on Conjecture 1.9 has been focused on explicit classes of gerbes, all of which have trivial bands. Conjecture 1.9 has been verified for trivial gerbes [9] and for certain gerbes over \mathbb{P}^1 -orbifolds [40]. A version of Conjecture 1.9 for genus 0 Gromov-Witten theory has also been proven for root gerbes over complex projective manifolds [10]. The case of toric gerbes is treated in [11]. All these works use very different methods.

In this paper, we provide more supporting evidence to Conjecture 1.9 by proving it for the G -gerbe $BH \rightarrow BQ$ obtained from a finite group extension. This is done in Section 6. Our approach starts with an isomorphism, which we deduce from Theorem 1.2, between the quantum cohomology ring of BH and the c -twisted quantum cohomology ring of \widehat{BH} . The isomorphism between quantum cohomology rings yields an equality between 3-point genus 0 Gromov-Witten invariants of \mathcal{Y} and 3-point genus 0 c -twisted Gromov-Witten invariants of $\widehat{\mathcal{Y}}$. We then apply a reconstruction-type argument, similar to the one used in [38, Section 4] to prove an equality between all Gromov-Witten invariants. See (6.16)-(6.17). Conjecture 1.9 then follows easily. See Section 6 for details.

To the best of our knowledge, our result is the first verification of Conjecture 1.9 for a class of gerbes with non-trivial bands.

In general, we believe that the ring isomorphism in Theorem 1.8 should yield the changes of variables needed in Conjecture 1.9.

1.8. Outlook. A natural question arising from our work is whether the claim (\star) can be generalized to G -gerbes for groups G which are not necessarily finite. On one hand, Mackey's machine on Lie groups is well studied in representation theory [31]. This suggests that the duality theorems proved in this paper may admit generalizations to more general G -gerbes. On the other hand, Mackey's machine on Lie groups is more involved than the simple case of finite groups presented in this paper. One can imagine that the complete picture of the duality theory for general G -gerbes will be more complicated, even for the trivial gerbe BG . We plan to study this more general theory in the near future.

1.9. Structure of the paper. The rest of this paper is organized as follows. In Section 2, we review some background material on groupoids, their extensions, gerbes, twisted sheaves, and Hochschild cohomology. In Section 3, we study the G -gerbe, $BH \rightarrow BQ$, arising from an extension of finite groups. In Section 4, we study cross-product algebras of a general G -gerbe and Hochschild cohomology. Section 5 is devoted to the study of the Chen-Ruan orbifold cohomology of a G -gerbe. The Gromov-Witten theory of a gerbe $BH \rightarrow BQ$ is treated in Section 6. In Section 7, we consider the category of sheaves on a G -gerbe.

Our study of the structure of the group algebra of a finite group extension yields some purely group-theoretic results. These results, which are discussed in Appendix A.1-A.2, may each be of independent interests.

Acknowledgment. We thank N. Higson and M. Rieffel for discussions on Mackey's machine. We thank Y. Ruan for his interests in this work and for his encouragement.

The results of this paper have been reported by the authors in several seminars and workshops held at University of Wisconsin-Madison, University of Utah, UC Riverside, MIT, University of British Columbia, University of Colorado-Boulder, National Taiwan University, and University of Göttingen.

X. T. is supported in part by NSF grant DMS-0703775 and 0900985. H.-H. T is supported in part by NSF grant DMS-0757722.

2. PRELIMINARY MATERIAL

In this section, we briefly review some background material about gerbes on orbifolds. We explain the basic concepts and tools that will be used in this paper.

2.1. Orbifolds and Groupoids. An orbifold is a separable Hausdorff topological space which is locally modeled on the quotient of \mathbb{R}^n by a linear action of a finite group. Such a topological space is very important in both mathematics and physics. In this paper, we study orbifolds in the viewpoint developed by Haefliger [35] and Moerdijk-Pronk [50]. Namely, we represent an orbifold as the quotient of a proper étale groupoid.

A groupoid is a small category all of whose morphisms are invertible. A groupoid \mathcal{G} is usually denoted by $\mathcal{G} \rightrightarrows \mathcal{G}_0$, where \mathcal{G}_0 is the set of objects of this category, and \mathcal{G} is the set of arrows of this category. Let m be the groupoid multiplication map $\mathcal{G} \times_{\mathcal{G}_0} \mathcal{G} \rightarrow \mathcal{G}$, i be the inverse on \mathcal{G} , s and t be the source and target maps $\mathcal{G} \rightarrow \mathcal{G}_0$, and u be the unit map $\mathcal{G}_0 \rightarrow \mathcal{G}$. An arrow $g \in \mathcal{G}$ is sometimes denoted by $x \rightarrow y$, which means that $s(g) = x$ and $t(g) = y$. A groupoid $\mathcal{G} \rightrightarrows \mathcal{G}_0$ is called a Lie groupoid if \mathcal{G} and \mathcal{G}_0 are smooth manifolds,⁵ all the structure maps

$$\mathcal{G} \times_{\mathcal{G}_0} \mathcal{G} \xrightarrow{m} \mathcal{G} \xrightarrow{i} \mathcal{G} \xrightarrow{s} \mathcal{G}_0 \xrightarrow{u} \mathcal{G},$$

are smooth, and the s and t maps are submersions. An étale groupoid is a special type of Lie groupoid if both the source maps s and the target maps t are local diffeomorphisms. A groupoid \mathcal{G} is proper if the map $(s, t) : \mathcal{G} \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is a proper map. A groupoid \mathcal{G} naturally defines an equivalence relation on the set, \mathcal{G}_0 , of objects. Two points x and y in \mathcal{G}_0 are equivalent if there is an arrow $g : x \rightarrow y$ in \mathcal{G} whose source is x and target is y . We use $[\mathcal{G}_0/\mathcal{G}]$ to denote the quotient space with respect to the above-defined equivalence relation. Moerdijk and Pronk [50] proved that any orbifold \mathcal{X} can be represented by the quotient space of a proper étale Lie groupoid \mathcal{G} . This leads to the following definition. More detailed discussions can be found in [5] and [50].

Definition 2.1. An orbifold groupoid is a proper étale groupoid \mathcal{G} , and an orbifold is the quotient space $[\mathcal{G}_0/\mathcal{G}]$ of an orbifold groupoid.

Locally, an orbifold \mathcal{X} is a quotient of \mathbb{R}^n by a linear action of a finite group Γ . This also shows that, locally, \mathcal{X} can be represented by the quotient space of the transformation groupoid $\mathbb{R}^n \rtimes \Gamma \rightrightarrows \mathbb{R}^n$. Gluing the local charts of an orbifold together, we obtain a proper étale groupoid, $\mathcal{G} \rightrightarrows \mathcal{G}_0$, representing \mathcal{X} . One quickly observes that the choice of charts on an orbifold is not unique. This allows an orbifold to be represented by different proper étale groupoids. However, a careful study of such a situation shows that these different étale groupoids are all Morita equivalent. The definition of Morita equivalence will be recalled in the next subsection. A crucial property of Morita equivalent groupoids is that the corresponding quotient spaces are isomorphic. In general, an orbifold can be uniquely represented by a Morita equivalence class of proper étale groupoids.

Given a groupoid \mathcal{G} , we can consider the space of “loops” in \mathcal{G} , which is defined to be

$$\mathcal{G}^{(0)} := \{g \in \mathcal{G} : s(g) = t(g)\}.$$

There is a natural \mathcal{G} -action on $\mathcal{G}^{(0)}$. Let $p : \mathcal{G}^{(0)} \rightarrow \mathcal{G}_0$ be the map defined by taking the source (=target) of an element $g \in \mathcal{G}^{(0)}$. The \mathcal{G} action on $\mathcal{G}^{(0)}$ is a map $\rho : \mathcal{G} \times_{t, \mathcal{G}_0, p} \mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(0)}$ such that $\rho(h, g) := hgh^{-1}$. If \mathcal{G} is a proper étale groupoid, one can easily check that the action groupoid

$$\mathcal{G} \ltimes \mathcal{G}^{(0)} \rightrightarrows \mathcal{G}^{(0)}$$

is also proper étale. This is called the inertia groupoid associated to the groupoid \mathcal{G} . If \mathcal{X} is the orbifold represented by \mathcal{G} , its inertia orbifold $I\mathcal{X}$ is represented by the inertia groupoid $\mathcal{G} \ltimes \mathcal{G}^{(0)}$.

The inertia groupoid has a natural cyclic structure $\theta : \mathcal{G}^{(0)} \rightarrow \mathcal{G} \ltimes \mathcal{G}^{(0)}$ defined by $\theta(g) := (g, g)$. This is very useful in the study of the cyclic theory [27] of the groupoid \mathcal{G} .

⁵ \mathcal{G} may not be Hausdorff.

Every element g in the loop space $\mathcal{G}^{(0)}$ acts on the tangent space $T_{p(g)}\mathcal{G}_0$. Assume that $\mathcal{G}^{(0)}$ is equipped with a \mathcal{G} -invariant almost complex structure. This makes $T_{p(g)}\mathcal{G}_0$ a complex vector space. Since g is of finite order, $T_{p(g)}\mathcal{G}_0$ splits into a sum of eigenspaces of the g action, i.e.,

$$T_{p(g)}\mathcal{G}_0 = \bigoplus_{k=0}^{r-1} V_k,$$

where g acts on V_k with eigenvalue $\exp(\frac{2\pi\sqrt{-1}k}{r})$. Define the age function on $\mathcal{G}^{(0)}$ by

$$(2.1) \quad \text{age}(g) := \sum_{k=0}^{r-1} \frac{k}{r} \dim(V_k) \in \mathbb{Q}.$$

One easily checks that age is a locally constant function on $\mathcal{G}^{(0)}$ and invariant under the \mathcal{G} action. Therefore the age function descends to a function on the inertia orbifold $I\mathcal{X}$, which is an important part of the definition of the Chen-Ruan orbifold cohomology [23].

2.2. Gerbes on orbifolds and groupoid extensions. The notion of a gerbe was introduced by Giraud [33] in algebraic geometry during his study of nonabelian cohomology. Let G be a topological group. In most of the cases of this paper, G is a finite group equipped with the discrete topology. The notion of a G -gerbe is a generalization of a principal G -bundle. Let BG be the classifying orbifold of the group G . A G -gerbe over a topological space X is a principal BG bundle over X . See [15] for related discussions. In this paper, we follow the groupoid approach to stacks and gerbes developed by Brylinski [18], Behrend, Xu [13], and Laurent-Gengoux, Stienon, and Xu [44].

Let \mathcal{B} be an orbifold. Then a G -gerbe \mathcal{X} over \mathcal{B} can be represented by a *groupoid G -extension*, which is a diagram

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{i} & \mathcal{H} & \xrightarrow{j} & \mathcal{Q} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathcal{G}_0 & \xlongequal{\quad} & \mathcal{H}_0 & \xlongequal{\quad} & \mathcal{Q}_0. \end{array}$$

In the above diagram,

- (1) \mathcal{G} is a principal G -bundle over \mathcal{G}_0 .
- (2) \mathcal{G}_0 , \mathcal{H}_0 , and \mathcal{Q}_0 are identical smooth manifolds.
- (3) i and j are smooth morphisms of Lie groupoids.
- (4) i is injective and j is surjective.
- (5) the groupoid $\mathcal{Q} \rightrightarrows \mathcal{Q}_0$ is a proper étale groupoid representing the orbifold \mathcal{B} .

We follow [44] and use

$$\mathcal{H} \rightarrow \mathcal{Q} \rightrightarrows \mathcal{Q}_0$$

to denote a groupoid extension of $\mathcal{Q} \rightrightarrows \mathcal{Q}_0$.

Definition 2.2. A G -gerbe groupoid over a groupoid $\mathcal{Q} \rightrightarrows \mathcal{Q}_0$ is a groupoid extension of the groupoid $\mathcal{Q} \rightrightarrows \mathcal{Q}_0$ such that its kernel is a locally trivial bundle of groups with fibers isomorphic to G .

Just like an orbifold has many different representations by proper étale groupoids, the above groupoid extension representation of a G -gerbe is, in general, not unique. A notion of Morita equivalence between groupoid extensions, which we now recall, was introduced by Laurent-Gengoux, Stienon, and Xu [44].

Let $\mathcal{G} \rightrightarrows \mathcal{G}_0$ and $\mathcal{H} \rightrightarrows \mathcal{H}_0$ be two groupoids. A Morita morphism from $\mathcal{G} \rightrightarrows \mathcal{G}_0$ to $\mathcal{H} \rightrightarrows \mathcal{H}_0$ is a smooth morphism (ϕ_1, ϕ_0) of Lie groupoids from $\mathcal{G} \rightrightarrows \mathcal{G}_0$ to $\mathcal{H} \rightrightarrows \mathcal{H}_0$

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\phi_1} & \mathcal{H} \\ \Downarrow & & \Downarrow \\ \mathcal{G}_0 & \xrightarrow{\phi_0} & \mathcal{H}_0 \end{array}$$

such that ϕ_0 is a surjective submersion and the pull-back of the groupoid $\mathcal{H} \rightrightarrows \mathcal{H}_0$ along the map ϕ_0 is isomorphic to $\mathcal{G} \rightrightarrows \mathcal{G}_0$. Two groupoids $\mathcal{G} \rightrightarrows \mathcal{G}_0$ and $\mathcal{H} \rightrightarrows \mathcal{H}_0$ are Morita equivalent if there is a third groupoid $\mathcal{K} \rightrightarrows \mathcal{K}_0$ together with Morita morphisms from $\mathcal{K} \rightrightarrows \mathcal{K}_0$ to both $\mathcal{G} \rightrightarrows \mathcal{G}_0$ and $\mathcal{H} \rightrightarrows \mathcal{H}_0$. A Morita morphism from the groupoid extension $Y_1 \rightarrow X_1 \rightrightarrows M_1$ to $Y_2 \rightarrow X_2 \rightrightarrows M_2$ consists of Morita morphisms $F_X : (X_1 \rightrightarrows M_1) \rightarrow (X_2 \rightrightarrows M_2)$ and $F_Y : (Y_1 \rightrightarrows M_1) \rightarrow (Y_2 \rightrightarrows M_2)$ such that the diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{F_Y} & Y_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{F_X} & X_2 \\ \Downarrow & & \Downarrow \\ M_1 & \xrightarrow{F_X = F_Y} & M_2 \end{array}$$

commutes. Two groupoid extensions $Y_i \rightarrow X_i \rightrightarrows M_i$, $i = 1, 2$, are Morita equivalent if there is a groupoid extension $Y \rightarrow X \rightrightarrows M$ together with Morita morphisms from $Y \rightarrow X \rightrightarrows M$ to both groupoid extensions.

An isomorphism class of G -gerbes over an orbifold \mathcal{B} is determined by a Morita equivalence class of G -groupoid extensions. For example, if Q is a finite group, a G -gerbe over an orbifold $[pt/Q]$ is represented by a group extension

$$1 \rightarrow G \rightarrow H \rightarrow Q \rightarrow 1.$$

One easily checks that two group extensions are Morita equivalent, as defined above, if and only if they are isomorphic group extensions (see, e.g., [58] for more discussions on group extensions). So isomorphism classes of G -gerbes over $[pt/Q]$ are in one-to-one correspondence with isomorphism classes of group extensions of Q by G .

2.3. \mathcal{G} -sheaves and modules of the groupoid algebra. An approach to sheaf theory on orbifolds via orbifold groupoids is explained in [49]. Let \mathcal{G} be a proper étale groupoid representing an orbifold $\mathcal{X} = [\mathcal{G}_0/\mathcal{G}]$. Denote by $\pi : \mathcal{G}_0 \rightarrow \mathcal{X}$ the projection from \mathcal{G}_0 to \mathcal{X} . A \mathcal{G} -sheaf is a sheaf \mathcal{S} on \mathcal{G}_0 with a \mathcal{G} action, i.e., for any $g \in \mathcal{G}$, $g : x \rightarrow y$ induces a morphism \hat{g} on stalks from \mathcal{S}_y to \mathcal{S}_x . \mathcal{G} -sheaves of abelian groups form an abelian category $\text{Sh}(\mathcal{G})$. A section ξ of a \mathcal{G} -sheaf is called invariant if $\hat{g}(\xi_y) = \xi_x$ for any $g : x \rightarrow y$. The functor

$$\Gamma_{\text{inv}} : \text{Sh}(\mathcal{G}) \ni \mathcal{S} \mapsto \{\text{invariant sections of } \mathcal{S}\}$$

is a left exact functor from $\text{Sh}(\mathcal{G})$ to the category Ab of abelian groups. The right derived functors of Γ_{inv} define the groupoid cohomology groups $H^\bullet(\mathcal{G}, \mathcal{S})$. If we consider compactly supported sections of a \mathcal{G} sheaf,

$$\Gamma_{\text{cpt}} : \text{Sh}(\mathcal{G}) \ni \mathcal{S} \mapsto \{\xi \in \Gamma_{\text{inv}}(\mathcal{S}) : \text{supp}(\xi) \text{ is compact}\}$$

also defines a left exact functor from $\text{Sh}(\mathcal{G})$ to the category Ab . Its right derived functor defines compactly supported cohomology groups $H_{\text{cpt}}^\bullet(\mathcal{G}, \mathcal{S})$.

If $\tilde{\mathcal{S}}$ is a sheaf on the orbifold \mathcal{X} , then the pullback $\pi^*(\tilde{\mathcal{S}})$ along the projection map π defines a \mathcal{G} -sheaf over \mathcal{G}_0 . Furthermore, if \mathcal{S} is a \mathcal{G} -sheaf, $\pi_!(\mathcal{S})$ defines a sheaf on \mathcal{X} , where $\pi_!(\mathcal{S})_x := H_{\text{cpt}}^\bullet(x/\pi, \pi_x^{-1}(\mathcal{S}))$ and x/π is the subgroupoid of \mathcal{G} over x . These two maps define natural functors between the category of \mathcal{G} -sheaves and the category of sheaves on \mathcal{X} . Let \mathcal{A} be a \mathcal{G} -sheaf of unital algebras. Define the convolution algebra $\mathcal{A} \rtimes \mathcal{G}$ to be the vector space $\Gamma_{\text{cpt}}(\mathcal{G}, s^*\mathcal{A})$ with the product

$$[a_1 * a_2](g) = \sum_{g_1 g_2 = g} [a_1](g_1) g_1([a_2](g_2)),$$

for any $a_1, a_2 \in \Gamma_{\text{cpt}}(\mathcal{G}, s^*\mathcal{A})$ and $g_1, g_2, g \in \mathcal{G}$. For example, when \mathcal{A} is the sheaf \mathcal{C}^∞ of smooth functions on \mathcal{G}_0 , we recover the standard groupoid algebra, which is $C_c^\infty(\mathcal{G})$ with the multiplication

$$(a_1 * a_2)(g) = \sum_{g_1 g_2 = g} a_1(g_1) a_2(g_2),$$

for any $a_1, a_2 \in C_c^\infty(\mathcal{G})$ and $g_1, g_2, g \in \mathcal{G}$. We will always identify the groupoid algebra $C_c^\infty(\mathcal{G})$ with the crossed product algebra $\mathcal{C}^\infty \rtimes \mathcal{G}$.

If \mathcal{S} is a \mathcal{G} -sheaf of vector spaces, then $\Gamma_{\text{cpt}}(\mathcal{S})$ is a module of the groupoid algebra $C_c^\infty(\mathcal{G})$ via

$$a\xi(x) = \sum_{t(g)=x} a(g)g(\xi(x)), \quad a \in C_c^\infty(\mathcal{G}), \xi \in \Gamma_{\text{cpt}}(\mathcal{S}), x \in \mathcal{G}_0.$$

This construction defines a natural additive functor from the category of \mathcal{G} -sheaves to the category of modules of $C_c^\infty(\mathcal{G})$.

Let c be a $U(1)$ -valued groupoid 2-cocycle on \mathcal{G} , i.e., $c : \mathcal{G} \times_{\mathcal{G}_0} \mathcal{G} \rightarrow U(1)$ such that

$$c(g_1, g_2)c(g_1 g_2, g_3) = c(g_1, g_2 g_3)c(g_2, g_3).$$

A c -twisted \mathcal{G} -sheaf is a sheaf over \mathcal{G}_0 together with a \mathcal{G} action such that for $g_1, g_2 \in \mathcal{G}$, the actions $\hat{g}_1 : \mathcal{S}_{x_1} \rightarrow \mathcal{S}_{x_2}$ and $\hat{g}_2 : \mathcal{S}_{x_2} \rightarrow \mathcal{S}_{x_3}$ satisfy

$$\hat{g}_1 \circ \hat{g}_2 = c(g_1, g_2) \widehat{g_1 g_2}.$$

The collection of c -twisted \mathcal{G} -sheaves forms an additive category. Let \mathcal{X} be the orbifold represented by \mathcal{G} . Then the cocycle c defines a $U(1)$ -gerbe, which we still denote by c , on the orbifold \mathcal{X} . We can consider c -twisted sheaves on \mathcal{X} as in [20] and [45]. One can easily check that the functors π^* and $\pi_!$ define natural functors between the category of c -twisted \mathcal{G} -sheaves and the category of c -twisted sheaves on \mathcal{X} .

Given a locally constant $U(1)$ -valued 2-cocycle c on \mathcal{G} and a \mathcal{G} -sheaf \mathcal{A} of unital algebras, we can define a c -twisted crossed product algebra $\mathcal{A} \rtimes_c \mathcal{G}$ to be the space $\Gamma_{\text{cpt}}(\mathcal{G}, s^*\mathcal{A})$ with the product defined by

$$(2.2) \quad [a_1 *_c a_2]_g = \sum_{g_1 g_2 = g} c(g_1, g_2) [a_1]_{g_1} g_1([a_2]_{g_2}), \quad a_1, a_2 \in \Gamma_{\text{cpt}}(\mathcal{G}, s^*\mathcal{A}), g_1, g_2, g \in \mathcal{G}.$$

When we take \mathcal{A} to be the sheaf \mathcal{C}^∞ of smooth functions, we have defined the c -twisted groupoid algebra $\mathcal{C}^\infty \rtimes_c \mathcal{G}$. Like the case without the twist, we have natural additive functors between the category of c -twisted \mathcal{G} -sheaves and category of modules of the c -twisted groupoid $\mathcal{C}^\infty \rtimes_c \mathcal{G}$.

Remark 2.3. For \mathcal{C}^∞ , the same formula in Equation (2.2) defines an associative algebra $\mathcal{C}^\infty \rtimes_c \mathcal{G}$ for any general $U(1)$ -valued 2-cocycle on \mathcal{G} even without the “locally constant” assumption. If we change the cocycle c by a coboundary, then a direct computation shows that the c -twisted crossed product algebras $\mathcal{C}^\infty \rtimes_c \mathcal{G}$ is changed by an isomorphism constructed with the coboundary. This shows that the isomorphism class of the c -twisted groupoid algebras only depends on the cohomology class of c in $H^2(\mathcal{G}, U(1)) = H^2(\mathcal{X}, U(1)) = H^3(\mathcal{X}, \mathbb{Z})$.

Remark 2.4. A $U(1)$ -valued 2-cocycle on \mathcal{G} can be used to define an S^1 -extension Γ of the groupoid \mathcal{G} ,

$$S^1 \rightarrow \Gamma \rightarrow \mathcal{G} \rightrightarrows \mathcal{G}_0.$$

The group $U(1) = S^1$ acts on the groupoid algebra of Γ by algebra automorphisms. Following [68], we consider the S^1 -invariant part $C_c^\infty(\Gamma)^{S^1}$, which is a subalgebra of the groupoid algebra $C_c^\infty(\Gamma)$. By direct computations, we can easily identify $C_c^\infty(\Gamma)^{S^1}$ with the c -twisted groupoid algebra $\mathcal{C}^\infty \rtimes_c \mathcal{G}$.

Assume that there is a \mathcal{G} -invariant symplectic form on \mathcal{G}_0 . Then via Fedosov's method, the first author in [64] constructed a \mathcal{G} -sheaf $\mathcal{A}^{((\hbar))}$ of deformation quantization of \mathcal{G}_0 . The crossed product $\mathcal{A}^{((\hbar))} \rtimes \mathcal{G}$ is a deformation of the groupoid algebra $\mathcal{C}^\infty \rtimes \mathcal{G}$. When c is locally constant, one can use Eqn. (2.2) to define a c -twisted $\mathcal{A}^{((\hbar))} \rtimes_c \mathcal{G}$ deformed groupoid algebra. More concretely, when c is locally constant, one has a natural flat connection on the associated S^1 extension $\Gamma \rightarrow \mathcal{G} \rightrightarrows \mathcal{G}_0$. Such a flat connection is sufficient to define a $U(1)$ -invariant deformation $\mathcal{A}^{((\hbar))}(\Gamma)$ of the groupoid algebra $C_c^\infty(\Gamma)$ by [64, Section 3]. It is not hard to check that the S^1 -invariant component $\mathcal{A}^{((\hbar))}(\Gamma)^{S^1}$ defines a deformation of the twisted groupoid algebra $C_c^\infty(\Gamma)^{S^1} \cong \mathcal{C}^\infty \rtimes_c \mathcal{G}$, whose product is written out like Eqn. (2.2).

2.4. Hochschild and cyclic cohomology. Let A be a unital algebra over the field \mathbb{C} or $\mathbb{C}((\hbar))$. In this paper, following [56], we will always work with bornological vector spaces, bornological tensor products between bornological vector spaces, bornological algebras, and modules of bornological algebras. As is explained in [53] and [56, Appendix A], the category of modules of a bornological algebra has enough projectives, and one can apply standard homological tools to define Hochschild and cyclic (co)homology of a bornological algebra. Let A^e be the enveloping algebra of A defined⁶ by $A^e := A \hat{\otimes} A^{op}$. For an A -bimodule M , define the Hochschild homology $HH_\bullet(A, M)$ and cohomology $HH^\bullet(A, M)$ by

$$HH_\bullet(A, M) := \mathrm{Tor}_\bullet^{A^e}(A, M), \quad HH^\bullet(A, M) := \mathrm{Ext}_{A^e}^\bullet(A, M).$$

We point out that the Yoneda product defines a natural product structure on $HH^\bullet(A, A)$. Therefore, $HH^\bullet(A, A)$ is an associative algebra.

If we consider the Bar-resolution of A as an A -bimodule, then we can write down explicit complexes $(C_\bullet(A, M), b)$ and $(C^\bullet(A, M), d)$ computing $HH_\bullet(A, M)$ and $HH^\bullet(A, M)$. Define $C_\bullet(A, M) = M \hat{\otimes} A^{\hat{\otimes} \bullet}$ together with $b : C_\bullet(A, M) \rightarrow C_{\bullet-1}(A, M)$ by

$$b(m \otimes a_1 \otimes \cdots \otimes a_k) = ma_1 \otimes a_2 \otimes \cdots \otimes a_k - m \otimes a_1 a_2 \otimes \cdots \otimes a_k + \cdots + (-1)^k a_k m \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

Define $C^\bullet(A, M) := \mathrm{Hom}(A^{\hat{\otimes} \bullet}, M)$ with $d : C^\bullet(A, M) \rightarrow C^{\bullet+1}(A, M)$ by

$$d\varphi(a_1, \dots, a_{k+1}) := a_1 \varphi(a_2, \dots, a_{k+1}) - \varphi(a_1 a_2, \dots, a_{k+2}) + \cdots + (-1)^{k+1} \varphi(a_1, \dots, a_k) a_{k+1}.$$

Let \mathbb{K} be a field. The cyclic homology of an algebra A over \mathbb{K} is computed by Connes' b - B bicomplex. For simplicity, we will consider the normalized complex $\overline{C}_\bullet(A) = A \hat{\otimes} (A/\mathbb{K})^{\hat{\otimes} \bullet}$ and $\overline{C}^\bullet(A) := \mathrm{Hom}(\overline{C}_\bullet(A), \mathbb{K})$ with the same differential b and d . Define $\overline{B} : \overline{C}_\bullet(A) \rightarrow \overline{C}_{\bullet+1}(A)$ by

$$\overline{B}(a_0 \otimes a_1 \otimes \cdots \otimes a_k) := \sum_i (-1)^{ik} 1 \otimes a_i \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

⁶In this paper, without causing extra confusion, we will use \otimes to denote the bornological tensor product between bornological spaces.

We can form the (b, \overline{B}) -bicomplex

$$\begin{array}{ccccc}
 \cdots & & \cdots & & \cdots \\
 \downarrow b & & \downarrow b & & \downarrow b \\
 \overline{C}_2(A) & \xleftarrow{\overline{B}} & \overline{C}_1(A) & \xleftarrow{\overline{B}} & \overline{C}_0(A) \\
 \downarrow b & & \downarrow b & & \\
 \overline{C}_1(A) & \xleftarrow{\overline{B}} & \overline{C}_0(A) & & \\
 \downarrow b & & & & \\
 \overline{C}_0(A) & & & &
 \end{array}$$

The homology of the total complex is equal to the cyclic homology of A . We can define the cyclic cohomology of A using a similar bicomplex.

The cyclic (co)homology of the groupoid algebra $C_c^\infty(\mathcal{G})$ for a proper étale groupoid \mathcal{G} was first computed by Brylinski and Nistor [19](also, see Crainic [27]) to be

$$HP_\bullet(C_c^\infty(\mathcal{G})) = H_{\text{cpt}}^\bullet(I\mathcal{X}).$$

If c is a $U(1)$ -valued 2-cocycle on \mathcal{G} , the cyclic (co)homology of the c -twisted groupoid algebra was computed by Tu and Xu [67] to be

$$HP_\bullet(C^\infty \rtimes_c \mathcal{G}) = H_{\text{cpt}}^\bullet(I\mathcal{X}, c).$$

The cohomology groups $H_{\text{cpt}}^\bullet(-)$ (and $H_{\text{cpt}}^\bullet(-, c)$) are compactly supported (c -twisted) cohomology groups of an orbifold defined by compactly supported differential forms.

When \mathcal{G}_0 is equipped with a \mathcal{G} -invariant symplectic form, the Hochschild (co)homology and cyclic (co)homology of the algebra $\mathcal{A}^{((h))} \rtimes \mathcal{G}$ were computed in the works [54] and [56] to be

$$HH^\bullet(\mathcal{A}^{((h))} \rtimes \mathcal{G}, \mathcal{A}^{((h))} \rtimes \mathcal{G}) = H^{\bullet-\ell}(I\mathcal{X})((\hbar)), \quad HP_\bullet(\mathcal{A}^{((h))} \rtimes \mathcal{G}) = H_{\text{cpt}}^\bullet(I\mathcal{X})((\hbar)).$$

2.5. Morita equivalence. Two algebras A and B are Morita equivalent if there is an additive equivalence between the category of A modules and the category of B modules. When both A and B are abelian, A is Morita equivalent to B if and only if A is isomorphic to B . Such a concept becomes very interesting in the study of noncommutative algebras. For example, for any nonzero integer n , the algebra of $n \times n$ matrices over a field \mathbb{K} is Morita equivalent to the field \mathbb{K} .

The additive equivalence between the categories of modules in the definition of Morita equivalence can be realized by A - B bimodules. Two algebras A and B are Morita equivalent if there is an A - B bimodule M and a B - A bimodule N such that as an A - A bimodule $M \otimes_B N$ is isomorphic to A and as a B - B bimodule $N \otimes_A M$ is isomorphic to B .

The equivalence functor preserves exact sequences and projective modules. Hence, Morita equivalent algebras have isomorphic Hochschild and cyclic (co)homologies and also K -groups. See [47].

Morita theory is generalized to bornological algebras [56, Appendix]. Similar results are naturally extended. For examples, Morita equivalent bornological algebras have isomorphic Hochschild and cyclic (co)homologies.

3. GROUP EXTENSION AND MACKEY MACHINE

In this section, we study the local model of the duality theorems that we are interested in. Namely, we look at the case when there is no space direction, and, therefore, a G -gerbe over an orbifold $[pt/Q] =$

BQ , where G and Q are both finite groups. As explained in Section 2.2, such a gerbe can be presented by a group extension

$$1 \longrightarrow G \longrightarrow H \longrightarrow Q \longrightarrow 1.$$

Different presentations of the same gerbe over $[pt/Q]$ are Morita equivalent, and correspond to isomorphic extensions. We refer readers to [44] for more details.

3.1. Group algebra. The results in this subsection are standard in group theory. We collect them here for later use. Let G, H, Q be finite groups that fit into the following exact sequence,

$$(3.1) \quad 1 \longrightarrow G \xrightarrow{i} H \xrightarrow{j} Q \longrightarrow 1.$$

We are interested in the structure of the group algebra $\mathbb{C}H$. We would like to understand it by using the information of G, Q , and the above exact sequence (3.1). The study of representations of the group H in terms of representations of G and Q goes back to Frobenius, Schur, and Clifford. In the case of continuous groups, this approach is often called the *Mackey machine*. We shall adapt this terminology in this paper.

We start by choosing a section $s : Q \rightarrow H$ of the group homomorphism j in the exact sequence (3.1) such that $j \circ s = id$, and $s(1) = 1$. Since G and Q are finite groups, such a section s always exists. In general, there are many possible choices of s , which later lead to equivalent structures.

It is important to point out that if the extension does not split, i.e., H is not isomorphic to a semi-direct product $G \rtimes Q$ as a group, the section s fails to be a group homomorphism. The failure of s being a group homomorphism is measured by

$$\tau : Q \times Q \rightarrow G, \quad \tau(q_1, q_2) := s(q_1)s(q_2)s(q_1q_2)^{-1}.$$

Note that $j(\tau(q_1, q_2)) = j(s(q_1))j(s(q_2))j(s(q_1q_2)^{-1}) = q_1q_2(q_1q_2)^{-1} = 1$, which shows that $\tau(q_1, q_2) \in \ker(j) = G$.

We rewrite the defining property of τ as

$$(3.2) \quad s(q_1)s(q_2) = \tau(q_1, q_2)s(q_1q_2).$$

Applying the above equation to $(s(q_1)s(q_2))s(q_3) = s(q_1)(s(q_2)s(q_3))$ yields

$$(3.3) \quad \tau(q_1, q_2)\tau(q_1q_2, q_3) = s(q_1)\tau(q_2, q_3)s(q_1)^{-1}\tau(q_1, q_2q_3).$$

Note that equations (3.2) and (3.3) define a G -gerbe BH over BQ .

With the above choice of the section s , the group H is isomorphic to $G \times Q$ as a set via the map

$$\alpha : H \rightarrow G \times Q, \quad \alpha(h) := (hs(j(h))^{-1}, j(h)).$$

The inverse of α is given by sending (g, q) to $i(g)s(q)$. Via the isomorphism α , the group structure on H defines a new group structure \cdot on $G \times Q$ given by

$$(3.4) \quad \begin{aligned} (g_1, q_1) \cdot (g_2, q_2) &= \alpha\left(\alpha^{-1}((g_1, q_1))\alpha^{-1}((g_2, q_2))\right) = \alpha(i(g_1)s(q_1)i(g_2)s(q_2)) \\ &= \alpha(i(g_1)s(q_1)i(g_2)s(q_1)^{-1}\tau(q_1, q_2)s(q_1q_2)) = (g_1 \operatorname{Ad}_{s(q_1)}(g_2)\tau(q_1, q_2), q_1q_2), \end{aligned}$$

where we use $\operatorname{Ad}_h(\cdot)$ to denote the conjugation action of an element $h \in H$ on G as a normal subgroup of H . Let

$$G \rtimes_{s, \tau} Q$$

denote the set $G \times Q$ with the above group structure (3.4) induced by H . Therefore, α is a group isomorphism from H to $G \rtimes_{s, \tau} Q$. Note that different choices of the section s define isomorphic group structures on $G \times Q$.

The group isomorphism α defines a natural isomorphism of group algebras

$$\mathbb{C}H \xrightarrow{\simeq} \mathbb{C}(G \rtimes_{s,\tau} Q),$$

which we still denote by α . With the data s and τ , the group algebra of $G \rtimes_{s,\tau} Q$ can be written as a twisted crossed product algebra $\mathbb{C}G \rtimes_{s,\tau} Q$, where an element $q \in Q$ acts on $\mathbb{C}G$ via conjugation by $s(q)$, and the failure of the action to be a group homomorphism is governed by τ .

To have a better understanding of the group algebra $\mathbb{C}H$, we look at the group algebra $\mathbb{C}G$. As a finite group, up to isomorphisms G has only finitely many irreducible unitary representations. Let \widehat{G} be the set⁷ of isomorphism classes of irreducible G representations. Furthermore, for every element $[\rho]$ in \widehat{G} , we fix a choice of an irreducible representation in the class $[\rho]$ denoted by

$$\rho : G \rightarrow \text{End}(V_\rho),$$

where V_ρ is a certain finite dimensional vector space. It is well-known (see, e.g., [32, Proposition 3.29]) that the group algebra $\mathbb{C}G$ is isomorphic to a direct sum of matrix algebra $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho)$,

$$\mathbb{C}G \xrightarrow{\simeq} \bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho),$$

via the natural map $\beta : g \mapsto (\rho(g))_{[\rho] \in \widehat{G}}$.

Our goal in the rest of this subsection is to replace the group algebra $\mathbb{C}G$ in the twisted crossed product $\mathbb{C}G \rtimes_{s,\tau} Q$ by the matrix algebras $\bigoplus_{[\rho] \in \widehat{G}} \text{End } V_\rho$. We need some preparations before proceeding to the explicit identification.

Let $\rho : G \rightarrow \text{End}(V_\rho)$ be an irreducible G representation. For $q \in Q$, consider the G representation $\tilde{\rho}$ defined by

$$G \ni g \mapsto \rho(\text{Ad}_{s(q)}(g)).$$

It is easy to see that $\tilde{\rho}$ is again an irreducible representation of G . If we consider isomorphism classes of irreducible G representations, the above construction defines a right Q -action on \widehat{G} ; namely, $q \in Q$ sends the class $[\rho] \in \widehat{G}$ to the class $[\tilde{\rho}] \in \widehat{G}$. This is a well-defined Q -action because conjugations by elements in G preserve isomorphism classes of irreducible G -representations. For convenience, we will write this right action as a left action. The image of the isomorphism class $[\rho] \in \widehat{G}$ under the action by q will be denoted by $q([\rho])$. By abuse of notation, the chosen irreducible G -representation that represents the class $q([\rho])$ will also be denoted by $q([\rho]) : G \rightarrow \text{End}(V_{q([\rho])})$.

Since the representation $q([\rho]) : G \rightarrow \text{End}(V_{q([\rho])})$ is, by definition, equivalent to the representation $\tilde{\rho} : G \rightarrow \text{End}(V_\rho)$ defined by $g \mapsto \rho(\text{Ad}_{s(q)}(g))$, there exists an isomorphism of vector spaces,

$$T_q^{[\rho]} : V_\rho \rightarrow V_{q([\rho])},$$

such that

$$\rho(\text{Ad}_{s(q)}(g)) = T_q^{[\rho]-1} \circ q([\rho])(g) \circ T_q^{[\rho]}.$$

To simplify our computation, we will always fix $T_1^{[\rho]}$ to be the identity map on V_ρ .

⁷If G is abelian, \widehat{G} is the Pontryagin dual group of G . In this paper we only need \widehat{G} to be a set.

We compute $\rho(\text{Ad}_{s(q_1)}(\text{Ad}_{s(q_2)}(g)))$. Using equation (3.3), we have

$$\begin{aligned}
& T_{q_1}^{[\rho]-1} \circ T_{q_2}^{q_1([\rho])^{-1}} \circ q_1 q_2([\rho])(g) \circ T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} \\
&= T_{q_1}^{[\rho]-1} \circ T_{q_2}^{q_1([\rho])^{-1}} \circ q_2(q_1([\rho]))(g) \circ T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} \\
&= T_{q_1}^{[\rho]-1} \circ q_1([\rho])(\text{Ad}_{s(q_2)}(g)) \circ T_{q_1}^{[\rho]} \\
&= \rho(\text{Ad}_{s(q_1)}(\text{Ad}_{s(q_2)}(g))) \\
&= \rho(\text{Ad}_{s(q_1)s(q_2)}(g)) \\
&= \rho(\text{Ad}_{\tau(q_1, q_2)s(q_1 q_2)}(g)) \\
&= \rho(\tau(q_1, q_2)) \circ \rho(\text{Ad}_{s(q_1 q_2)}(g)) \circ \rho(\tau(q_1, q_2))^{-1} \\
&= \rho(\tau(q_1, q_2)) \circ T_{q_1 q_2}^{[\rho]-1} \circ q_1 q_2(\rho)(g) \circ T_{q_1 q_2}^{[\rho]} \circ \rho(\tau(q_1, q_2))^{-1}
\end{aligned}$$

It follows that $T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} \circ \rho(\tau(q_1, q_2)) \circ T_{q_1 q_2}^{[\rho]-1}$, as a map in $\text{End}(V_{q_1 q_2([\rho])})$, commutes with the representation $q_1 q_2([\rho])$. By Schur's lemma, there must be a constant $c^{[\rho]}(q_1, q_2)$ such that $T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} \circ \rho(\tau(q_1, q_2)) \circ T_{q_1 q_2}^{[\rho]-1}$ is $c^{[\rho]}(q_1, q_2)$ times the identity map. In other words,

$$(3.5) \quad T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} = c^{[\rho]}(q_1, q_2) T_{q_1 q_2}^{[\rho]} \rho(\tau(q_1, q_2))^{-1}.$$

When we require the family $\{V_\rho\}$ to consist of unitary representations, the operator $T_q^{[\rho]}$ can also be chosen to be unitary. Therefore, $c^{[\rho]}(q_1, q_2)$ actually takes value in $U(1)$.

Proposition 3.1. *The function*

$$c : \widehat{G} \times Q \times Q \rightarrow U(1), \quad ([\rho], q_1, q_2) \mapsto c^{[\rho]}(q_1, q_2)$$

is a 2-cocycle on the groupoid $\widehat{G} \rtimes Q$ such that $c^{[\rho]}(1, q) = c^{[\rho]}(q, 1) = 1$ for any $[\rho] \in \widehat{G}$, $q \in Q$. The cohomology class defined by c is independent of the choices of the section s and the operator $T_q^{[\rho]}$.

Proof. Consider the composition of maps $T_{q_3}^{q_1 q_2([\rho])} \circ T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]}$. By associativity, we have

$$(3.6) \quad (T_{q_3}^{q_1 q_2([\rho])} \circ T_{q_2}^{q_1([\rho])}) \circ T_{q_1}^{[\rho]} = T_{q_3}^{q_1 q_2([\rho])} \circ (T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]}).$$

Using equation (3.5), we compute the left hand side of (3.6) to be

$$\begin{aligned}
& c^{q_1([\rho])}(q_2, q_3) T_{q_2 q_3}^{q_1([\rho])} q_1([\rho])(\tau(q_2, q_3))^{-1} \circ T_{q_1}^{[\rho]} \\
&= c^{q_1([\rho])}(q_2, q_3) T_{q_2 q_3}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} \circ T_{q_1}^{[\rho]-1} q_1([\rho])(\tau(q_2, q_3))^{-1} \circ T_{q_1}^{[\rho]} \\
&= c^{q_1([\rho])}(q_2, q_3) c^{[\rho]}(q_1, q_2 q_3) T_{q_1 q_2 q_3}^{[\rho]} \rho(\tau(q_1, q_2 q_3))^{-1} \circ T_{q_1}^{[\rho]-1} q_1([\rho])(\tau(q_2, q_3))^{-1} \circ T_{q_1}^{[\rho]} \\
&= c^{q_1([\rho])}(q_2, q_3) c^{[\rho]}(q_1, q_2 q_3) T_{q_1 q_2 q_3}^{[\rho]} \rho(\tau(q_1, q_2 q_3))^{-1} \rho(\text{Ad}_{s(q_1)}(\tau(q_2, q_3)^{-1})) \\
&= c^{q_1([\rho])}(q_2, q_3) c^{[\rho]}(q_1, q_2 q_3) T_{q_1 q_2 q_3}^{[\rho]} \rho(\text{Ad}_{s(q_1)}(\tau(q_2, q_3)) \tau(q_1, q_2 q_3))^{-1}.
\end{aligned}$$

Similarly, using equation (3.5), we compute the right hand side of equation (3.6),

$$\begin{aligned}
& T_{q_3}^{q_1 q_2([\rho])} \circ T_{q_1 q_2}^{[\rho]} c^{[\rho]}(q_1, q_2) \rho(\tau(q_1, q_2))^{-1} \\
&= T_{q_1 q_2 q_3}^{[\rho]} c^{[\rho]}(q_1 q_2, q_3) c^{[\rho]}(q_1, q_2) \rho(\tau(q_1 q_2, q_3))^{-1} \rho(\tau(q_1, q_2))^{-1} \\
&= T_{q_1 q_2 q_3}^{[\rho]} c^{[\rho]}(q_1 q_2, q_3) c^{[\rho]}(q_1, q_2) \rho(\tau(q_1, q_2) \tau(q_1 q_2, q_3))^{-1}.
\end{aligned}$$

From the above computation and equation (3.3), we obtain the cocycle equation

$$(3.7) \quad c^{q_1([\rho])}(q_2, q_3) c^{[\rho]}(q_1, q_2 q_3) = c^{[\rho]}(q_1 q_2, q_3) c^{[\rho]}(q_1, q_2).$$

Note that by equation (3.2) and $s(1) = 1$, we have $\tau(q, 1) = \tau(1, q) = 1$. As $T_1^{[\rho]} = 1$ for any $[\rho] \in \widehat{G}$, we conclude from equation (3.5) that $c^{[\rho]}(1, q) = c^{[\rho]}(q, 1) = 1$.

One easily computes that changing the section s to a new section s' leads to changing the isomorphism $T_q^{[\rho]}$ to $T_q^{[\rho]} \circ \rho(s'(q)s(q)^{-1})$. By Schur's lemma, one can prove that different choices of the isomorphism $T_q^{[\rho]}$ are different by a scalar in $U(1)$. By straightforward computations and the above observations, one can show that, up to a coboundary, the cocycle c is independent of the choices of the section s and $T_q^{[\rho]}$. \square

Motivated by the structures associated to $\{T_q^{[\rho]} | q \in Q, [\rho] \in \widehat{G}\}$ and c , we consider the space

$$\oplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \otimes \mathbb{C}Q.$$

This space is spanned by elements of the form (x_ρ, q) , where x_ρ is an element in $\text{End}(V_\rho)$ with $[\rho] \in \widehat{G}$ and $q \in Q$. We define the product \circ between (x_{ρ_1}, q_1) and $(\tilde{x}_{\rho_2}, q_2)$ by

$$(x_{\rho_1}, q_1) \circ (\tilde{x}_{\rho_2}, q_2) := \begin{cases} (x_{\rho_1} T_{q_1}^{[\rho_1]-1} \tilde{x}_{q_1([\rho_1])} T_{q_1}^{[\rho_1]} \rho_1(\tau(q_1, q_2)), q_1 q_2), & \text{if } [\rho_2] = q_1([\rho_1]), \\ 0 & \text{otherwise.} \end{cases}$$

We check the associativity of the product \circ . It is sufficient to check the identity

$$\begin{aligned} & [(x_\rho, q) \circ (y_{q([\rho])}, q')] \circ (z_{qq'([\rho])}, q'') \\ &= (x_\rho T_q^{[\rho]-1} y_{q([\rho])} T_q^{[\rho]} \rho(\tau(q, q')), qq') \circ (z_{qq'([\rho])}, q'') \\ &= (x_\rho T_q^{[\rho]-1} y_{q([\rho])} T_q^{[\rho]} \rho(\tau(q, q')) T_{qq'}^{[\rho]-1} z_{qq'([\rho])} T_{qq'}^{[\rho]} \rho(\tau(qq', q'')), qq'q'') \\ &= (x_\rho T_q^{[\rho]-1} y_{q([\rho])} T_{q'}^{q'([\rho])^{-1}} c^{[\rho]}(q, q') z_{qq'([\rho])} T_{qq'}^{[\rho]} \rho(\tau(qq', q'')), qq'q'') \\ &= (x_\rho T_q^{[\rho]-1} y_{q([\rho])} T_{q'}^{q'([\rho])^{-1}} z_{qq'([\rho])} T_{q'}^{q([\rho])} T_q^{[\rho]} \rho(\tau(q, q') \tau(qq', q'')), qq'q'') \\ &= (x_\rho T_q^{[\rho]-1} y_{q([\rho])} T_{q'}^{q'([\rho])^{-1}} z_{qq'([\rho])} T_{q'}^{q([\rho])} q([\rho]) (\tau(q', q'')) T_q^{[\rho]} \rho(\tau(q, q'q'')), qq'q'') \\ &= (x_\rho, q) \circ [(y_{q([\rho])}, q') \circ (z_{qq'([\rho])}, q'')], \end{aligned}$$

where in the second and third equalities, we used equation (3.5) and the fact that c takes value in $U(1)$, and in the fourth equality, we used the cocycle condition (3.3) for τ .

The space $\oplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \otimes \mathbb{C}Q$ with the product structure introduced above will be denoted by $\oplus_{[\rho]} \text{End}(V_\rho) \rtimes_{T, \tau} Q$, and will be called the twisted crossed product algebra.

Proposition 3.2. *The map $\chi : G \times Q \ni (g, q) \mapsto \sum_{[\rho] \in \widehat{G}} (\rho(g), q)$ defines an algebra isomorphism from the group algebra $\mathbb{C}G \rtimes_{s, \tau} Q$ to the twisted crossed product algebra $\oplus_{[\rho]} \text{End}(V_\rho) \rtimes_{T, \tau} Q$. Hence, $\chi \circ \alpha : \mathbb{C}H \rightarrow \oplus_{[\rho]} \text{End}(V_\rho) \rtimes_{T, \tau} Q$ is an algebra isomorphism.*

Proof. We first prove that χ is an isomorphism of vector spaces. Since both algebras are finite dimensional with the same dimension, it suffices to prove that χ is injective. If $\sum_g \sum_q c_{g,q}(g, q) \in \mathbb{C}G \rtimes_{s, \tau} Q$ is in the kernel of χ , then as the map $\mathbb{C}G \rightarrow \oplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho)$ is an isomorphism, we see that for any fixed q , $\sum_g c_{g,q}(g, q)$ must be zero. Hence, $c_{g,q} = 0$ for any g, q . Hence, $\sum_g \sum_q c_{g,q}(g, q) = 0$. This shows that the kernel of χ is trivial, and, therefore, χ is an isomorphism of vector spaces.

We are left to show that χ is an algebra homomorphism. It suffices to prove

$$\chi((g_1, q_1)) \circ \chi((g_2, q_2)) = \chi((g_1, q_2) \cdot (g_2, q_2)).$$

By the definition of χ , we have

$$\begin{aligned}
& \chi((g_1, q_1)) \circ \chi((g_2, q_2)) \\
&= \sum_{[\rho]} (\rho(g_1), q_1) \circ \sum_{[\rho']} (\rho'(g_2), q_2) \\
&= \sum_{[\rho]} (\rho(g_1), q_1) \circ (q_1([\rho])(g_2), q_2) \\
&= \sum_{[\rho]} (\rho(g_1) T_{q_1}^{[\rho_1]-1} q_1([\rho])(g_2) T_{q_1}^{[\rho_1]} \rho(\tau(q_1, q_2)), q_1 q_2) \\
&= \sum_{[\rho]} (\rho(g_1) \rho(\text{Ad}_{s(q_1)}(g_2)) \rho(\tau(q_1, q_2)), q_1 q_2) \\
&= \sum_{[\rho]} (\rho(g_1 s(q_1) g_2 s(q_1)^{-1} \tau(q_1, q_2)), q_1 q_2) \\
&= \chi((g_1 \text{Ad}_{s(q_1)}(g_2) \tau(q_1, q_2), q_1 q_2)) \\
&= \chi((g_1, q_1) \cdot (g_2, q_2)).
\end{aligned}$$

□

3.2. Mackey machine and Morita equivalence. In this subsection, we introduce the framework that will be used to explain the Mackey machine on representations of groups. Our treatment is essentially a reformulation of the Mackey machine in the language of Morita equivalence of algebras. Such a reformulation seems to be known among experts [31]. However, we are unable to locate any good references that provide the exact statements we need.

For an extension of finite groups $1 \rightarrow G \rightarrow H \rightarrow Q \rightarrow 1$ considered in Section 3.1, the Mackey machine provides a way to describe a representation of H via representations of G and Q . Such an idea goes back to Frobenius and Schur and was first explicitly explained by Clifford in [25]. In our language, the Mackey machine may be formulated as saying that the representation theory of the group H is equivalent to the representation theory of the transformation groupoid $\widehat{G} \rtimes Q$ together with the $U(1)$ -valued groupoid cocycle c , which is introduced in equation (3.5).

Our arguments of the above equivalence go as follows. It is known that the category of representations of the group H is equivalent to the category of representations of the group algebra $\mathbb{C}H$. On the other hand, given the groupoid $\widehat{G} \rtimes Q$ with the $U(1)$ -valued groupoid cocycle c , let

$$C(\widehat{G} \rtimes Q, c)$$

be the twisted groupoid algebra associated to the cocycle c on $\widehat{G} \rtimes Q$, as introduced in [68] (see Section 2.3 for a review). We spell out the definition of $C(\widehat{G} \rtimes Q, c)$. As a space $C(\widehat{G} \rtimes Q, c)$ consists of $C(\widehat{G})$ -valued functions on Q , i.e., $C(\widehat{G} \rtimes Q, c)$ consists of functions on $\widehat{G} \times Q$. For $([\rho], q) \in \widehat{G} \times Q$ we abuse notation and denote by $([\rho], q)$ the function on $\widehat{G} \times Q$ which takes value 1 at the point $([\rho], q)$ and 0 elsewhere. The collection $\{([\rho], q)\}$ of functions on $\widehat{G} \times Q$ form a basis of $C(\widehat{G} \rtimes Q, c)$. The product on $C(\widehat{G} \rtimes Q, c)$ is defined by

$$([\rho], q) \circ ([\rho'], q') = \begin{cases} c^{[\rho]}(q, q')([\rho], qq') & \text{if } [\rho'] = q([\rho]) \\ 0 & \text{otherwise} \end{cases}.$$

The associativity of the above product \circ follows from the cocycle condition (3.7) of c .

Our formulation of the Mackey machine is the following theorem.

Theorem 3.3. *The group algebra $\mathbb{C}H$ is Morita equivalent to the twisted groupoid algebra $C(\widehat{G} \rtimes Q, c)$.*

Proof. By Proposition 3.2, the group algebra $\mathbb{C}H$ is isomorphic to the algebra $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \rtimes_{T, \tau} Q$. Set $A = \bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \rtimes_{T, \tau} Q$ and $B = C(\widehat{G} \rtimes Q, c)$. To prove the theorem, we will construct a left A - B bimodule M and a B - A bimodule N , such that, as an A - A bimodule, $M \otimes_B N$ is isomorphic to A , and as a B - B bimodule, $N \otimes_A M$ is isomorphic to B .

The A - B bimodule M , as vector space, is isomorphic to $\bigoplus_{[\rho] \in \widehat{G}} V_\rho \times Q$. The space $\bigoplus_{[\rho] \in \widehat{G}} V_\rho \times Q$ is spanned by elements of the form (ξ_ρ, q) with $\xi_\rho \in V_\rho$ and $q \in Q$. The left A -module structure on M is defined by

$$(x_{\rho_0}, q_0)(\xi_\rho, q) := \begin{cases} (c^{[\rho_0]}(q_0, q)x_{\rho_0}T_{q_0}^{[\rho_0]-1}(\xi_{q_0([\rho_0])}), q_0q) & \text{if } [\rho] = q_0([\rho_0]) \\ 0 & \text{otherwise.} \end{cases}$$

We check that this is a left A -module structure on M :

$$\begin{aligned} & (x_{\rho_1}, q_1)((x_{\rho_0}, q_0)(\xi_\rho, q)) \\ &= (x_{\rho_1}, q_1)(c^{[\rho_0]}(q_0, q)x_{\rho_0}T_{q_0}^{[\rho_0]-1}(\xi_{q_0([\rho_0])}), q_0q) \\ &= (x_{\rho_1}c^{[\rho_1]}(q_1, q_0q)c^{q_1([\rho_1])}(q_0, q)T_{q_1}^{[\rho_1]-1}x_{q_1([\rho_1])}T_{q_0}^{q_1([\rho_1])^{-1}}(\xi_{q_1q_0([\rho_1])}), q_1q_0q) \\ &= (x_{\rho_1}T_{q_1}^{[\rho_1]-1}x_{q_1([\rho_1])}T_{q_1}^{[\rho_1]}c^{[\rho_1]}(q_1q_0, q)T_{q_1}^{[\rho_1]-1}c^{[\rho_1]}(q_1, q_0)T_{q_0}^{q_1([\rho_1])^{-1}}(\xi_{q_1q_0([\rho_1])}), q_1q_0q) \\ &= (x_{\rho_1}T_{q_1}^{[\rho_1]-1}x_{q_1([\rho_1])}T_{q_1}^{[\rho_1]}c^{[\rho_1]}(q_1q_0, q)\rho_1(\tau(q_1, q_0))T_{q_1q_0}^{[\rho_1]}(\xi_{q_1q_0([\rho_1])}), q_1q_0q) \\ &= ((x_{\rho_1}, q_1)(x_{\rho_0}, q_0))(\xi_\rho, q), \end{aligned}$$

where in the third equality, we used the cocycle condition (3.7) of c , and in the fourth equality, we used equation (3.5).

The right B -module structure on M is defined by

$$(\xi_\rho, q)([\rho_1], q_1) := \begin{cases} (c^{[\rho]}(q, q_1)\xi_\rho, qq_1) & \text{if } [\rho_1] = q([\rho]) \\ 0 & \text{otherwise.} \end{cases}$$

We check that this is a right B -module structure on M :

$$\begin{aligned} & ((\xi_\rho, q)([\rho_1], q_1))([\rho_2], q_2) \\ &= (c^{[\rho]}(q, q_1)\xi_\rho, qq_1)([\rho_2], q_2) \\ &= (c^{[\rho]}(q, q_1)c^{[\rho]}(qq_1, q_2)\xi_\rho, qq_1q_2) \\ &= (c^{[\rho]}(q, q_1q_2)c^{q([\rho])}(q_1, q_2)\xi_\rho, qq_1q_2) \\ &= (\xi_\rho, q)((q([\rho]), q_1)(qq_1([\rho]), q_2)), \end{aligned}$$

where in the third equality, we used the cocycle condition (3.7) of c .

We check that the left A -module and the right B -module structures commute:

$$\begin{aligned}
& (x_{\rho_1}, q_1)((\xi_\rho, q)([\rho_2], q_2)) \\
&= (x_{\rho_1}, q_1)(c^{[\rho]}(q, q_2)\xi_\rho, qq_2) \\
&= (c^{[\rho_1]}(q_1, qq_2)x_{\rho_1}T_{q_1}^{[\rho_1]-1}(c^{q_1([\rho_1])}(q, q_2)\xi_{q_1(\rho_1)}), q_1qq_2) \\
&= (c^{[\rho_1]}(q_1, qq_2)c^{q_1([\rho_1])}(q, q_2)x_{\rho_1}T_{q_1}^{[\rho_1]-1}(\xi_{q_1(\rho_1)}), q_1qq_2) \\
&= (c^{[\rho_1]}(q_1, q)c^{[\rho_1]}(q_1q, q_2)x_{\rho_1}T_{q_1}^{[\rho_1]-1}(\xi_{q_1(\rho_1)}), q_1qq_2) \\
&= ((x_{\rho_1}, q_2)(\xi_\rho, q))([\rho_2], q_2).
\end{aligned}$$

The B - A bimodule N is isomorphic to $\bigoplus_{[\rho] \in \widehat{G}} V_\rho^* \times Q$ as a vector space. So N is spanned by elements of the form (η_ρ, q) , where $q \in Q$ and $\eta_\rho \in V_\rho^*$ is a linear functional on V_ρ . The right A -module on N is defined by

$$(\eta_\rho, q)(x_{\rho_0}, q_0) = \begin{cases} (c^{[\rho_0]}(q_0, q_0^{-1}q)\eta_{\rho_0} \circ x_{\rho_0} \circ T_{q_0}^{[\rho_0]-1}, q_0^{-1}q) & \text{if } [\rho] = [\rho_0] \\ 0 & \text{otherwise.} \end{cases}$$

We check that this is a right A -module structure on N :

$$\begin{aligned}
& ((\eta_\rho, q)(x_{\rho_0}, q_0))(x_{\rho_1}, q_1) \\
&= (c^{[\rho_0]}(q_0, q_0^{-1}q)\eta_{\rho_0} \circ x_{\rho_0} \circ T_{q_0}^{[\rho_0]-1}, q_0^{-1}q)(x_{\rho_1}, q_1) \\
&= (c^{q_0^{-1}([\rho_1])}(q_0, q_0^{-1}q)c^{[\rho_1]}(q_1, q_1^{-1}q_0^{-1}q)\eta_{q_0^{-1}([\rho_1])} \circ x_{q_0^{-1}([\rho_1])} \circ T_{q_0}^{q_0^{-1}([\rho_1])^{-1}} \circ x_{\rho_1} \circ T_{q_1}^{[\rho_1]-1}, q_1^{-1}q_0^{-1}q) \\
&= (c^{q_0^{-1}([\rho_1])}(q_0, q_1)c^{q_0^{-1}([\rho_1])}(q_0q_1, q_1^{-1}q_0^{-1}q)\eta_{q_0^{-1}([\rho_1])} \circ x_{q_0^{-1}([\rho_1])} \circ T_{q_0}^{q_0^{-1}([\rho_1])^{-1}} \circ x_{\rho_1} \circ T_{q_1}^{[\rho_1]-1}, q_1^{-1}q_0^{-1}q) \\
&= (c^{q_0^{-1}([\rho_1])}(q_0q_1, q_1^{-1}q_0^{-1}q)\eta_{q_0^{-1}([\rho_1])} \circ x_{q_0^{-1}([\rho_1])} \circ T_{q_0}^{q_0^{-1}([\rho_1])^{-1}} \circ x_{\rho_1} \circ T_{q_0}^{q_0^{-1}([\rho_1])} \circ \\
&\quad \circ q_0^{-1}([\rho_1])(\tau(q_0, q_1)) \circ T_{q_0q_1}^{q_0^{-1}([\rho_1])^{-1}}, q_1^{-1}q_0^{-1}q) \\
&= (\eta_\rho, q)((x_{\rho_0}, q_0)(x_{\rho_1}, q_1)),
\end{aligned}$$

where in the third equality, we used the cocycle condition (3.7), and in the fourth equality we used the definition of the cocycle c , equation (3.5).

The left B -module structure on $N = \bigoplus_{[\rho] \in \widehat{G}} V_\rho^* \times Q$ is defined by

$$([\rho_0], q_0)(\eta_\rho, q) := \begin{cases} (c^{[\rho]}(qq_0^{-1}, q_0)\eta_\rho, qq_0^{-1}) & \text{if } [\rho_0] = qq_0^{-1}([\rho]) \\ 0 & \text{otherwise.} \end{cases}$$

We check that this is a left B -module structure:

$$\begin{aligned}
& ([\rho_1], q_1)([\rho_0], q_0)(\eta_\rho, q) \\
&= ([\rho_1], q_1)(c^{[\rho]}(qq_0^{-1}, q_0)\eta_\rho, qq_0^{-1}) \\
&= (c^{[\rho]}(qq_0^{-1}q_1^{-1}, q_1)c^{[\rho]}(qq_0^{-1}, q_0)\eta_\rho, qq_0^{-1}q_1^{-1}) \\
&= (c^{[\rho]}(qq_0^{-1}q_1^{-1}, q_1q_0)c^{q_0q_1^{-1}([\rho])}(q_1, q_0)\eta_\rho, qq_0^{-1}q_1^{-1}) \\
&= (\eta_\rho, q)(qq_0^{-1}q_1^{-1}([\rho]), q_1)(qq_0^{-1}[\rho], q_0),
\end{aligned}$$

where in the third equality, we used the cocycle condition (3.7) of c .

We check that the left B -module and the right A -module structures on N commute:

$$\begin{aligned}
& [(qq_0^{-1}([\rho]), q_0)(\eta_\rho, q)](x_\rho, q_1) \\
&= (c^{[\rho]}(qq_0^{-1}, q_0)\eta_\rho, qq_0^{-1})(x_\rho, q_1) \\
&= (c^{[\rho]}(qq_0^{-1}, q_0)c^{[\rho]}(q_1, q_1^{-1}qq_0^{-1})\eta_\rho \circ x_\rho \circ T_{q_1}^{[\rho]-1}, q_1^{-1}qq_0^{-1}) \\
&= (c^{[\rho]}(q_1, q_1^{-1}q)c^{q_1([\rho])}(q_1^{-1}qq_0^{-1}, q_0)\eta_\rho \circ x_\rho \circ T_{q_1}^{[\rho]-1}, q_1^{-1}qq_0^{-1}) \\
&= (qq_0^{-1}([\rho]), q_0)[(\eta_\rho, q)(x_\rho, q_1)]
\end{aligned}$$

where in the third equality, we used the cocycle condition of c .

Next we show that $M \otimes_B N \cong A$ as an A - A bimodule. Define a map $\Xi : M \otimes_{\mathbb{C}} N \rightarrow A$ by

$$\Xi((\xi_\rho, q), (\eta_{q'q'^{-1}([\rho])}, q')) := \begin{cases} (\xi_\rho \otimes \eta_{q'q'^{-1}([\rho])} \circ T_{q'q'^{-1}}^{\frac{c^{q'q'^{-1}([\rho])}(q'q'^{-1}, q)}{c^{q'q'^{-1}([\rho])}(q'q'^{-1}, q'q'^{-1})}}, q'q'^{-1}) & \text{if } q'q'^{-1}([\rho]) = [\rho'] \\ 0 & \text{otherwise.} \end{cases}$$

We check that

$$\begin{aligned}
& \Xi((\xi_\rho, q)(q([\rho]), q_0), (\eta_{qq_0q'^{-1}([\rho])}, q')) \\
&= \Xi((c^{[\rho]}(q, q_0)\xi_\rho, qq_0), (\eta_{qq_0q'^{-1}([\rho])}, q')) \\
&= (\frac{c^{[\rho]}(q, q_0)c^{qq_0q'^{-1}([\rho])}(q'q_0^{-1}q^{-1}, qq_0)}{c^{qq_0q'^{-1}([\rho])}(q'q_0^{-1}q^{-1}, qq_0q'^{-1})}\xi_\rho \otimes \eta_{qq_0q'^{-1}([\rho])} \circ T_{qq_0q'^{-1}}^{[\rho]}, qq_0q'^{-1}) \\
&= (\frac{c^{qq_0q'^{-1}([\rho])}(q'q_0^{-1}q^{-1}, q)c^{qq_0q'^{-1}([\rho])}(q'q_0^{-1}, q_0)}{c^{qq_0q'^{-1}([\rho])}(q'q_0^{-1}q^{-1}, qq_0q'^{-1})}\xi_\rho \otimes \eta_{qq_0q'^{-1}([\rho])} \circ T_{qq_0q'^{-1}}^{[\rho]}, qq_0q'^{-1}) \\
&= \Xi(\xi_\rho, q), (q([\rho]), q_0)(\eta_{qq_0q'^{-1}([\rho])}, q'),
\end{aligned}$$

where in the third equality, we used the cocycle condition (3.7) of c . Therefore, Ξ passes to define a map, which we still denote by Ξ , from $M \otimes_B N$ to A .

We check that the map Ξ is compatible with the left A -module structure:

$$\begin{aligned}
& (x_\rho, q)\Xi((\xi_{q([\rho])}, q_0), (\eta_{qq_0q_1^{-1}([\rho])}, q_1)) \\
&= (x_\rho, q)(\xi_{q([\rho])} \otimes \eta_{qq_0q_1^{-1}([\rho])} \circ T_{q_0q_1^{-1}}^{\frac{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0)}{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0q_1^{-1})}}, q_0q_1^{-1}) \\
&= (x_\rho \circ T_q^{[\rho]-1} \circ \xi_{q([\rho])} \otimes \eta_{qq_0q_1^{-1}([\rho])} \circ T_{q_0q_1^{-1}}^{\frac{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0)}{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0q_1^{-1})}} T_q^{[\rho]} \rho(\tau(q, q_0q_1^{-1})), qq_0q_1^{-1}) \\
&= (x_\rho \circ T_q^{[\rho]-1} \circ \xi_{q([\rho])} \otimes \eta_{qq_0q_1^{-1}([\rho])} \frac{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0)}{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0q_1^{-1})} \circ T_{q_0q_1^{-1}}^{\frac{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0)}{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0q_1^{-1})}} T_q^{[\rho]} \rho(\tau(q, q_0q_1^{-1})), qq_0q_1^{-1}) \\
&= (x_\rho \circ T_q^{[\rho]-1} \circ \xi_{q([\rho])} \otimes \eta_{qq_0q_1^{-1}([\rho])} \circ \frac{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0)c^{[\rho]}(q, q_0q_1^{-1})}{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}, q_0q_1^{-1})} T_{qq_0q_1^{-1}}^{[\rho]}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \Xi((x_\rho, q)(\xi_{q([\rho])}, q_0), (\eta_{qq_0q_1^{-1}([\rho])}, q_1)) \\
&= \Xi((x_\rho \circ T_q^{[\rho]-1} \circ \xi_{q([\rho])} c^{[\rho]}(q, q_0), qq_0), (\eta_{qq_0q_1^{-1}([\rho])}, q_1)) \\
&= (x_\rho \circ T_q^{[\rho]-1}(\xi_{q([\rho])}) \otimes \eta_{qq_0q_1^{-1}([\rho])} \circ \frac{c^{[\rho]}(q, q_0)c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}q^{-1}, qq_0)}{c^{qq_0q_1^{-1}([\rho])}(q_1q_0^{-1}q^{-1}, qq_0q_1^{-1})} T_{qq_0q_1^{-1}}^{[\rho]}).
\end{aligned}$$

Using the cocycle condition (3.7) of c , it is not difficult to check that

$$\frac{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0) c^{[\rho]}(q, q_0 q_1^{-1})}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1})} = \frac{c^{[\rho]}(q, q_0) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1} q^{-1}, q q_0)}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1} q^{-1}, q q_0 q_1^{-1})}.$$

We conclude that Ξ is compatible with the left A -module structure. Therefore, we have the following equality:

$$(x_\rho, q) \Xi((\xi_{q([\rho])}, q_0), (\eta_{q_0 q_1^{-1}([\rho])}, q_1)) = \Xi((x_\rho, q)(\xi_{q([\rho])}, q_0), (\eta_{q_0 q_1^{-1}([\rho])}, q_1)).$$

We check that Ξ is compatible with the right A -module structure:

$$\begin{aligned} & \Xi((\xi_\rho, q_0), (\eta_{q_0 q_1^{-1}([\rho])}, q_1))(x_{q_0 q_1^{-1}([\rho])}, q) \\ &= (\xi_\rho \otimes \eta_{q_0 q_1^{-1}([\rho])} \circ T_{q_0 q_1^{-1}}^{[\rho]} \frac{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1})}, q_0 q_1^{-1})(x_{q_0 q_1^{-1}([\rho])}, q) \\ &= (\frac{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1})} \xi_\rho \otimes \eta_{q_0 q_1^{-1}([\rho])} \circ T_{q_0 q_1^{-1}}^{[\rho]} T_{q_0 q_1^{-1}}^{[\rho]-1} x_{q_0 q_1^{-1}([\rho])} T_{q_0 q_1^{-1}}^{[\rho]} \rho(\tau(q_0 q_1^{-1}, q)), q_0 q_1^{-1} q) \\ &= (\frac{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1})} \xi_\rho \otimes \eta_{q_0 q_1^{-1}([\rho])} \circ x_{q_0 q_1^{-1}([\rho])} T_{q_0 q_1^{-1}}^{[\rho]} \rho(\tau(q_0 q_1^{-1}, q)), q_0 q_1^{-1} q) \\ &= (\frac{c^{[\rho]}(q_0 q_1^{-1}, q) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1})} \xi_\rho \otimes \eta_{q_0 q_1^{-1}([\rho])} \circ x_{q_0 q_1^{-1}([\rho])} T_q^{q_0 q_1^{-1}([\rho])^{-1}} T_{q_0 q_1^{-1} q}^{[\rho]} q, q_0 q_1^{-1} q). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \Xi((\xi_\rho, q_0), (\eta_{q_0 q_1^{-1}([\rho])}, q_1))(x_{q_0 q_1^{-1}([\rho])}, q) \\ &= \Xi((\xi_\rho, q_0), (c^{q_0 q_1^{-1}([\rho])}(q, q^{-1} q_1) \eta_{q_0 q_1^{-1}([\rho])} \circ x_{q_0 q_1^{-1}([\rho])} \circ T_q^{q_0 q_1^{-1}([\rho])^{-1}}, q^{-1} q_1)) \\ &= (\xi_\rho \otimes \eta_{q_0 q_1^{-1}([\rho])} \circ x_{q_0 q_1^{-1}([\rho])} T_q^{q_0 q_1^{-1}([\rho])^{-1}} T_{q_0 q_1^{-1} q}^{[\rho]} \frac{c^{q_0 q_1^{-1}([\rho])}(q, q^{-1} q_1) c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0 q_1^{-1} q)}, q_0 q_1^{-1} q). \end{aligned}$$

We need to show that

$$\frac{c^{[\rho]}(q_0 q_1^{-1}, q) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1})} = \frac{c^{q_0 q_1^{-1}([\rho])}(q, q^{-1} q_1) c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0)}{c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0 q_1^{-1} q)},$$

which is equivalent to

$$\begin{aligned} & c^{[\rho]}(q_0 q_1^{-1}, q) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0) c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0 q_1^{-1} q) \\ &= c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1}) c^{q_0 q_1^{-1}([\rho])}(q, q^{-1} q_1) c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0). \end{aligned}$$

Using the cocycle condition (3.7) of c , we have

$$\begin{aligned} & c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0 q_1^{-1} q) c^{[\rho]}(q_0 q_1^{-1}, q) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0) \\ &= c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0 q_1^{-1}) c^{q_0 q_1^{-1} q([\rho])}(q^{-1}, q) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0), \end{aligned}$$

and

$$\begin{aligned}
& c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1}) c^{q_0 q_1^{-1}([\rho])}(q, q^{-1} q_1) c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0) \\
&= c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0 q_1^{-1}) c^{q_0 q_1^{-1}([\rho])}(q, q^{-1} q_1 q_0^{-1}) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0) \\
&= c^{q_0 q_1^{-1}([\rho])}(q, q^{-1}) c^{q_0 q_1^{-1} q([\rho])}(q^{-1} q_1 q_0^{-1}, q_0 q_1^{-1}) c^{q_0 q_1^{-1}([\rho])}(q_1 q_0^{-1}, q_0).
\end{aligned}$$

To prove the above two expressions are equal, we are left to prove

$$c^{q_0 q_1^{-1} q([\rho])}(q^{-1}, q) = c^{q_0 q_1^{-1}([\rho])}(q, q^{-1}).$$

Using the fact that $c^{q_0 q_1^{-1}([\rho])}(1, q) = 1 = c^{q_0 q_1^{-1}([\rho])}(q, 1)$, the above equality is equivalent to

$$c^{q_0 q_1^{-1}([\rho])}(q, 1) c^{q_0 q_1^{-1} q([\rho])}(q^{-1}, q) = c^{q_0 q_1^{-1}([\rho])}(q, q^{-1}) c^{q_0 q_1^{-1}([\rho])}(1, q),$$

which again follows by the cocycle condition (3.7) of c .

In summary, we have shown that Ξ is an A - A bimodule map from $M \otimes_B N$ to A . From the definition of Ξ , we can see that the image of Ξ contains all elements of the form (x_ρ, q) , where x_ρ is a rank 1 operator on V_ρ for any $[\rho] \in \widehat{G}$, $q \in Q$. As these elements span the whole space of A , we conclude that Ξ is surjective. By counting dimensions of $M \otimes_B N$ and A , we conclude that Ξ must be an isomorphism.

Next, we define an isomorphism, $\Theta : N \otimes_A M \rightarrow B$, of B - B bimodules:

$$\Theta((\eta_\rho, q), (\xi_{\rho'}, q')) = \begin{cases} (\eta_\rho(\xi_\rho) \frac{1}{c^{[\rho]}(q, q^{-1} q')} q([\rho]), q^{-1} q') & \text{if } [\rho'] = [\rho] \\ 0 & \text{otherwise.} \end{cases}$$

Here, we use the notation $(a[\rho], q)$ to denote the function on $\widehat{G} \times Q$ that takes the value $a \in \mathbb{C}$ at $([\rho], q) \in \widehat{G} \times Q$ and 0 elsewhere.

We check that Θ is well-defined:

$$\begin{aligned}
& \Theta((\eta_{\rho_0}, q_0)(x_{\rho_0}, q), (\xi_{q([\rho_0])}, q_1)) \\
&= \Theta((c^{[\rho_0]}(q, q^{-1} q_0) \eta_{\rho_0} \circ x_{\rho_0} \circ T_q^{[\rho_0]-1}, q^{-1} q_0), (\xi_{q([\rho_0])}, q_1)) \\
&= (\frac{c^{[\rho_0]}(q, q^{-1} q_0)}{c^{q([\rho_0])}(q^{-1} q_0, q_0^{-1} q q_0)} \eta_{\rho_0} \circ x_{\rho_0} \circ T_q^{[\rho_0]-1}(\xi_{q([\rho_0])}) q_0([\rho_0]), q_0^{-1} q q_1) \\
&= \Theta((\frac{c^{[\rho_0]}(q, q^{-1} q_0) c^{[\rho_0]}(q_0, q_0^{-1} q q_1)}{c^{[\rho_0]}(q, q_1) c^{q([\rho_0])}(q^{-1} q_0, q_0^{-1} q q_0)} \eta_{\rho_0}, q_0), (x_{\rho_0}, q)(\xi_{q([\rho_0])}, q_1)) \\
&= \Theta((\eta_{\rho_0}, q_0), (x_{\rho_0}, q)(\xi_{q([\rho_0])}, q_1)),
\end{aligned}$$

where in the fourth equality, we used the cocycle condition (3.7) of c .

We check that Θ is compatible with the B - B bimodule structure. We first check the left B -module structure:

$$\begin{aligned}
& \Theta(([\rho], q)(\eta_{q q_0^{-1}([\rho])}, q_0), (\xi_{q q_0^{-1}([\rho])}, q_1)) \\
&= \Theta((c^{[\rho]}(q q_0^{-1}, q) \eta_{q q_0^{-1}([\rho])}, q_0 q^{-1}), (\xi_{q q_0^{-1}([\rho])}, q_1)) \\
&= (\frac{c^{q q_0^{-1}([\rho])}(q_0 q^{-1}, q)}{c^{q q_0^{-1}([\rho])}(q_0 q^{-1}, q q_0^{-1} q_1)} \eta_{q q_0^{-1}([\rho])}(\xi_{q q_0^{-1}([\rho])})[\rho], q q_0^{-1} q_1) \\
&= (\frac{c^{[\rho]}(q, q_0^{-1} q_1)}{c^{q q_0^{-1}([\rho])}(q_0, q_0^{-1} q_1)} \eta_{q q_0^{-1}([\rho])}(\xi_{q q_0^{-1}([\rho])})[\rho], q q_0^{-1} q_1) \\
&= ([\rho], q) \Theta((\eta_{q q_0^{-1}([\rho])}, q_0), (\xi_{q q_0^{-1}([\rho])}, q_1)),
\end{aligned}$$

where in the third equality, we used the cocycle condition (3.7) of c .

Compatibility with the right B -module structure is checked as follows:

$$\begin{aligned}
& \Theta((\eta_\rho, q_0), (\xi_\rho, q_1)(q_1([\rho]), q)) \\
&= \Theta((\eta_\rho, q_0), (c^{[\rho]}(q_1, q)\xi_\rho, q_1q)) \\
&= (\eta_\rho(\xi_\rho) \frac{c^{[\rho]}(q_1, q)}{c^{[\rho]}(q_0, q_0^{-1}q_1q)} q_0([\rho]), q_0^{-1}q_1q) \\
&= (\frac{c^{q_0([\rho])}(q_0^{-1}q_1, q)}{c^{[\rho]}(q_0, q_0^{-1}q_1)} \eta_\rho(\xi_\rho) q_0([\rho]), q_0^{-1}q_1q) \\
&= \Theta((\eta_\rho, q_0), (\xi_\rho, q_1))(q_1([\rho]), q),
\end{aligned}$$

where in the third equality, we used the cocycle condition (3.7) of c .

In summary, we have constructed a B - B bimodule map Θ from $N \otimes_A M$ to B . It is not difficult to check that the image of Θ contains all elements of the form $([\rho], q)$. Therefore, Θ is a surjective map. By dimension counting, we conclude that Θ must be an isomorphism. \square

Remark 3.1. We point out that with our definitions of the bimodules maps Ξ and Θ , one can easily check that

$$\begin{aligned}
& \Xi((\xi_\rho, q), (\eta_{\rho'}, q'))(\xi_{\rho''}, q'') = (\xi_\rho, q)\Theta((\eta_{\rho'}, q'), (\xi_{\rho''}, q'')), \\
& \Theta((\eta_\rho, q), (\xi_{\rho'}, q'))(\eta_{\rho''}, q'') = (\eta_\rho, q)\Xi((\xi_{\rho'}, q'), (\eta_{\rho''}, q'')).
\end{aligned}$$

This is crucial in the study of the centers of A and B in Section 3.3.

Since Hochschild cohomology and K -theory are both Morita invariants, we have the following:

Corollary 3.2. The Hochschild cohomology (respectively, K -theory) of the algebra $\mathbb{C}H$ is isomorphic to the Hochschild cohomology (respectively, K -theory) of $C(\widehat{G} \rtimes Q, c)$.

In the rest of this subsection, we study the twisted groupoid algebra $C(\widehat{G} \rtimes Q, c)$. Denote the set of orbits of the Q action on \widehat{G} by $\text{Orb}^Q(\widehat{G})$. The groupoid $\widehat{G} \rtimes Q$ decomposes into a disjoint union of groupoids, $\sqcup_{O_k \in \text{Orb}^Q(\widehat{G})} O_k \rtimes Q$, where each component, $O_k \rtimes Q$, is a subgroupoid of $\widehat{G} \rtimes Q$. The restriction of the $U(1)$ -valued cocycle c yields a $U(1)$ -valued cocycle c_k on the component $O_k \rtimes Q$. It is easy to see that the twisted groupoid algebra decomposes into a direct sum of subalgebras,

$$C(\widehat{G} \rtimes Q, c) = \bigoplus_{O_k \in \text{Orb}^Q(\widehat{G})} C(O_k \rtimes Q, c_k).$$

Fix an orbit O_k in $\text{Orb}^Q(\widehat{G})$ and consider the twisted groupoid algebra $C(O_k \rtimes Q, c_k)$. Choose a point $[\rho] \in O_k$ and let $\text{Stab}([\rho]) \subset Q$ be the stabilizer subgroup of $[\rho]$, which consists of $q \in Q$ such that $q([\rho]) = [\rho]$. The cocycle c restricts to define a $U(1)$ -valued cocycle

$$c_{[\rho]} : \text{Stab}([\rho]) \times \text{Stab}([\rho]) \rightarrow U(1)$$

on the group $\text{Stab}([\rho])$. Given these data, we consider the *twisted group algebra* $C(\text{Stab}([\rho]), c_{[\rho]})$. By definition, $C(\text{Stab}([\rho]), c_{[\rho]})$ is additively the same as the group algebra $\mathbb{C} \text{Stab}([\rho])$ but equipped with a twisted product:

$$q_1 \cdot q_2 := c_{[\rho]}(q_1, q_2)(q_1 q_2).$$

Discussions of twisted group algebras can be found in [7] and [41].

Theorem 3.3. The twisted groupoid algebra $C(O_k \rtimes Q, c_k)$ is Morita equivalent to the twisted group algebra $C(\text{Stab}([\rho]), c_{[\rho]})$.

Proof. We explain the geometric correspondent of this Morita equivalence in the case when the cocycle c is trivial. The transformation groupoid $O_k \rtimes Q$ is Morita equivalent to the group $\text{Stab}([\rho])$ with the equivalent bibundle $M_0 := \{([\rho], q) : q \in Q\}$. The groupoid $O_k \rtimes Q$ acts on M_0 from the right by right multiplication, and the group $\text{Stab}([\rho])$ acts on M_0 from the left by left multiplication. It is straightforward to check that M_0 defines a Morita equivalent bibundle between $\text{Stab}([\rho])$ and $O_k \rtimes Q$. The result of [52] shows that such an equivalent bibundle defines a Morita equivalence between the group algebra $\mathbb{C} \text{Stab}([\rho])$ and the groupoid algebra $C(O_k \rtimes Q)$.

Inspired by this, we write down a Morita equivalent $C(\text{Stab}([\rho]), c_{[\rho]})$ - $C(O_k \rtimes Q, c_k)$ bimodule. Let M be the space of functions on M_0 . Define the left $C(\text{Stab}([\rho]), c_{[\rho]})$ module structure on M by

$$q_0 \delta_{([\rho], q)} = c^{[\rho]}(q_0, q) \delta_{([\rho], q_0 q)}, \quad q_0 \in \text{Stab}([\rho]), q \in Q,$$

where $\delta_{([\rho], q)}$ denotes the function on M_0 taking the value 1 at $([\rho], q)$ and zero everywhere else. We define the right $C(O_k \rtimes Q, c_k)$ module on M by

$$\delta_{([\rho], q)}([\rho_0], q_0) := \begin{cases} c^\rho(q, q_0) \delta_{[\rho], q q_0} & \text{if } q([\rho]) = [\rho_0] \\ 0 & \text{otherwise.} \end{cases}$$

We can also write down a Morita equivalent $C(O_k \rtimes Q, c_k)$ - $C(\text{Stab}([\rho]), c_{[\rho]})$ bimodule N . Let N be the space of functions on the set $\{q^{-1}([\rho]), q : q \in Q\}$. Define the left $C(O_k \rtimes Q, c_k)$ module structure by

$$([\rho_0], q_0) \delta_{(q^{-1}([\rho]), q)} := \begin{cases} c^{[\rho_0]}(q_0, q) \delta_{(q_0^{-1} q^{-1}[\rho], q_0 q)} & \text{if } q_0([\rho]) = q^{-1}([\rho]) \\ 0 & \text{otherwise.} \end{cases}$$

We also define the right $C(\text{Stab}([\rho]), c_{[\rho]})$ module structure by

$$\delta_{(q^{-1}([\rho]), q)} q_0 := c^{q^{-1}([\rho])}(q, q_0) \delta_{(q^{-1}([\rho]), q q_0)}.$$

And we define the bimodule map $X : M \otimes_{C(O_k \rtimes Q, c_k)} N \rightarrow C(\text{Stab}([\rho]), c_{[\rho]})$ by

$$X(\delta_{([\rho], q_0)}, \delta_{(q_1^{-1}([\rho]), q_1)}) := \begin{cases} c^{[\rho]}(q_0, q_1) q_0 q_1 & \text{if } q_0([\rho]) = q_1^{-1}([\rho]) \\ 0 & \text{otherwise.} \end{cases}$$

We define the bimodule map $Y : N \otimes_{C(\text{Stab}([\rho]), c_{[\rho]})} M \rightarrow C(O \rtimes Q, c)$ by

$$Y(\delta_{(q_0^{-1}([\rho]), q_0)}, \delta_{([\rho], q_1)}) := c^{q_0^{-1}([\rho])}(q_0, q_1) (q_0^{-1}([\rho]), q_0 q_1).$$

The verification that the above data define Morita equivalent bimodules is routine and left to the reader. \square

In conclusion, we know that the category of representations of the group H is isomorphic to the category of representations of the sum of twisted group algebras $C(\text{Stab}([\rho]), c_{[\rho]})$, where the sum is taken over the set of Q -orbits in \widehat{G} and we choose an element $[\rho]$ from each Q -orbit. Representations of the $c_{[\rho]}$ -twisted group algebra correspond to projective representations of the group $\text{Stab}([\rho])$ with cocycle $c_{[\rho]}$. This is exactly what the Mackey machine [31] states about representation theory of a finite group H with a normal subgroup G .

3.3. Isomorphism between centers. According to Theorem 3.3, the group algebra $\mathbb{C}H$ is Morita equivalent to the twisted groupoid algebra $C(\widehat{G} \rtimes Q, c)$. It is known (see, e.g., [8]) that Morita equivalent unital algebras have isomorphic centers. This implies that the center of $\mathbb{C}H$ is isomorphic to the center of $C(\widehat{G} \rtimes Q, c)$. In the proof of Theorem 3.3, we constructed Morita equivalent bimodules between the two algebras. The goal of this subsection is to analyze the isomorphisms between the centers of $\mathbb{C}H$ and $C(\widehat{G} \rtimes Q, c)$ induced from these explicit bimodules.

Let $A = \mathbb{C}H$ and $B = C(\widehat{G} \rtimes Q, c)$. Let M (respectively, N) be the A - B (respectively, B - A) Morita equivalence bimodule as defined in the proof⁸ of Theorem 3.3. Using the isomorphism $\Xi : M \otimes_B N \rightarrow A$ (respectively, $\Theta : N \otimes_A M \rightarrow B$) constructed in the proof of Theorem 3.3, we know that the algebra $\text{End}_B(M)$ (respectively, $\text{End}_A(N)$) of linear endomorphisms of M (respectively, N) commuting with the action of B (respectively, A) is isomorphic to A (respectively, B) under the isomorphism Ξ (respectively, Θ).

We write out these isomorphisms more explicitly. For any $[\rho] \in \widehat{G}$, choose a basis ξ_i^ρ of V_ρ and a dual basis η_ρ^i of V_ρ^* such that $\eta_\rho^i(\xi_j^\rho) = \delta_j^i$ (the Kronecker delta function). Define two maps

$$\begin{aligned} \Psi : \text{End}_B(M) &\rightarrow A, & x &\mapsto \sum_{\rho, i} \Xi(x(\xi_i^\rho, 1), (\eta_\rho^i, 1)) \\ \Phi : \text{End}_A(N) &\rightarrow B, & y &\mapsto \sum_{\rho, i} \frac{1}{\dim(V_\rho)} \Theta(y(\eta_\rho^i, 1), (\xi_i^\rho, 1)). \end{aligned}$$

We explain the construction in more detail in the case of Ψ . As x acts on M and commutes with the B action, x defines a linear endomorphism on $M \otimes_B N$ commuting with the right A action. Ξ is an isomorphism from $M \otimes_B N$ to A as an A - A bimodule. Hence, under the isomorphism Ξ , x becomes a linear endomorphism of A commuting with the right A action. Such a linear endomorphism is naturally identified as an element in A by taking its value on the unit element of A . Applying this to x , we map x to $\Xi(x\Xi^{-1}(1))$. It is easy to check that $\sum_{\rho, i} (\xi_i^\rho, 1) \otimes (\eta_\rho^i, 1)$ is mapped to the unit under Ξ . This gives the above definition of Ψ . The exact same reasoning leads to the definition of Φ . It is not difficult to check that Ψ and Φ are identity maps when we restrict them to A and B . Therefore, the action of $\Psi(x)$, (respectively, $\Phi(y)$) as an element in A (respectively, B) on M (respectively, N) is identical to the action of x on M (respectively, y). From this, we can easily check that Ψ and Φ are algebra isomorphisms and inverses to each other.

Now we consider the application of the above maps to the centers of A and B . Let y be an element in the center of $A = \mathbb{C}H$. As y commutes with elements of A , y as an endomorphism on N is in $\text{End}_A(N)$. Therefore, under the map Φ , y is mapped to an element of B . Notice that $\Phi(y)$, as an element in B , acts on N in the same way as the action of y on N . Therefore, $\Phi(y)$ as an element in B acts on N and commutes with any element of B . Therefore, we conclude that $\Phi(y)$ must be in the center of B . A similar reasoning shows that if x is an element in the center of B , then $\Psi(x)$ is in the center of A . Thus, we have constructed two algebra isomorphisms, Φ and Ψ , that identify the centers of $\mathbb{C}H$ and $C(\widehat{G} \rtimes Q, c)$.

We study the compatibility of the above maps with respect to the center of the group algebra $\mathbb{C}Q$. Recall that $j : H \rightarrow Q$ is a group epimorphism. Therefore, j induces an algebra epimorphism from $\mathbb{C}H$ to $\mathbb{C}Q$. The center of $\mathbb{C}Q$ has a canonical basis indexed by the conjugacy classes of Q . Therefore, the center of $\mathbb{C}H$, as a vector space, decomposes into a direct sum of subspaces $Z(\mathbb{C}H)_{\langle q \rangle}$ indexed by conjugacy classes $\langle q \rangle$ of Q ,

$$Z(\mathbb{C}H) = \bigoplus_{\langle q \rangle \subset Q} Z(\mathbb{C}H)_{\langle q \rangle}.$$

We examine the center of $C(\widehat{G} \rtimes Q, c)$ more carefully. Recall that $\{([\rho], q)\}_{q \in Q, [\rho] \in \widehat{G}}$ forms a basis of $C(O_k \rtimes Q, c_k)$. If $f = \sum_{[\rho], q} a_{[\rho], q}([\rho], q)$ is in the center of $C(\widehat{G} \rtimes Q, c)$, then for any $([\rho_0], q_0) \in C(O_k \rtimes Q, c_k)$, we have

$$f([\rho_0], q_0) = ([\rho_0], q_0)f,$$

⁸We use Proposition 3.2 to identify $\mathbb{C}H$ with $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \rtimes_{T, \tau} Q$.

which is equivalent to

$$\sum_q a_{q^{-1}([\rho_0]),q} c^{q^{-1}([\rho_0])}(q, q_0)(q^{-1}([\rho_0]), qq_0) = \sum_q a_{q_0([\rho_0]),q} c^{[\rho_0]}(q_0, q)([\rho_0], q_0q).$$

Replacing q by $q_0qq_0^{-1}$ in the left hand side of the above equality, we obtain

$$\begin{aligned} \sum_q a_{q_0q^{-1}q_0^{-1}([\rho_0]),q_0qq_0^{-1}} c^{q_0q^{-1}q_0^{-1}([\rho_0])}(q_0q^{-1}q_0^{-1}, q_0q)(q_0q^{-1}q_0^{-1}([\rho_0]), q_0q) \\ = \sum_q a_{q_0([\rho_0]),q} c^{[\rho_0]}(q_0, q)([\rho_0], q_0q). \end{aligned}$$

Comparing the two sides of the above equation, we see that, for every $q \in Q$,

$$a_{q_0q^{-1}q_0^{-1}([\rho_0]),q_0qq_0^{-1}} c^{q_0q^{-1}q_0^{-1}([\rho_0])}(q_0q^{-1}q_0^{-1}([\rho_0]), q_0q) = a_{q_0([\rho_0]),q} c^{[\rho_0]}(q_0, q)([\rho_0], q_0q).$$

This implies the following results.

- (1) If q is such that $q_0q^{-1}q_0^{-1}([\rho_0]) \neq [\rho_0]$ (equivalently $q(q_0([\rho_0])) \neq q_0([\rho_0])$), then

$$a_{q_0q^{-1}q_0^{-1}([\rho_0]),q_0qq_0^{-1}} = a_{q_0([\rho_0]),q} = 0.$$

- (2) If q is such that $q_0q^{-1}q_0^{-1}([\rho_0]) = [\rho_0]$ (equivalently $q(q_0([\rho_0])) = q_0([\rho_0])$), then

$$a_{q_0q^{-1}q_0^{-1}([\rho_0]),q_0qq_0^{-1}} c^{q_0q^{-1}q_0^{-1}([\rho_0])}(q_0q^{-1}q_0^{-1}, q_0q) = a_{q_0([\rho_0]),q} c^{[\rho_0]}(q_0, q).$$

As $q(q_0([\rho_0])) = q_0([\rho_0])$, the above equality is equivalent to

$$a_{[\rho_0],q_0qq_0^{-1}} c^{[\rho_0]}(q_0q^{-1}q_0^{-1}, q_0) = a_{q_0([\rho_0]),q} c^{[\rho_0]}(q_0, q),$$

which is equivalent to (replacing $q_0([\rho_0])$ by $[\rho_0]$)

$$a_{q_0^{-1}([\rho_0]),q_0qq_0^{-1}} c^{q_0^{-1}([\rho_0])}(q_0q^{-1}q_0^{-1}, q_0) = a_{[\rho_0],q} c^{q_0^{-1}([\rho_0])}(q_0, q).$$

In summary, we can write an element f in the center of $C(\widehat{G} \rtimes Q, c)$ as

$$(3.8) \quad f = \sum_q \sum_{[\rho], q([\rho])=[\rho]} a_{[\rho],q}([\rho], q),$$

such that

$$(3.9) \quad a_{[\rho_0],q} = a_{q_0^{-1}([\rho_0]),q_0qq_0^{-1}} \frac{c^{q_0^{-1}([\rho_0])}(q_0q^{-1}q_0^{-1}, q_0)}{c^{q_0^{-1}([\rho_0])}(q_0, q)}.$$

By equation (3.9), f can be written as

$$f = \sum_{\langle q \rangle \subset Q} \sum_{q_0 \in \langle q \rangle} \sum_{[\rho], q_0([\rho])=[\rho]} a_{[\rho],q_0}([\rho], q_0),$$

such that every component

$$\sum_{q_0 \in \langle q \rangle} \sum_{[\rho], q_0([\rho])=[\rho]} a_{[\rho],q_0}([\rho], q_0)$$

is in the center of $C(\widehat{G} \rtimes Q, c)$. This shows that the center $Z(C(\widehat{G} \rtimes Q, c))$ decomposes into a direct sum of subspaces $Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$ indexed by conjugacy classes of Q ,

$$Z(C(\widehat{G} \rtimes Q, c)) = \bigoplus_{\langle q \rangle \subset Q} Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}.$$

The map $\chi \circ \alpha$ constructed in Proposition 3.2 defines an isomorphism of algebras from $\mathbb{C}H$ to $\bigoplus_{[\rho]} \text{End}(V_\rho) \rtimes_{T, \tau} Q$. Composing this isomorphism with the above map Φ , we obtain an algebra isomorphism from the center of $\mathbb{C}H$ to the center of $C(\widehat{G} \rtimes Q, c)$. We denote this map by I .

Proposition 3.4. *The isomorphism $I : Z(\mathbb{C}H) \rightarrow Z(C(\widehat{G} \rtimes Q, c))$ is compatible with the decompositions into subspaces indexed by conjugacy classes of Q , i.e., I is an isomorphism from $Z(\mathbb{C}H)_{\langle q \rangle}$ to $Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$.*

An application of Proposition 3.4 is discussed in Appendix A.1.

Proof. We prove that the isomorphism I maps the subspace $Z(\mathbb{C}H)_{\langle q_0 \rangle}$, indexed by the conjugacy class $\langle q \rangle$ of Q , into the subspace of $Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$ with the same index. Then, by the fact that I is an isomorphism, we know that I must be an isomorphism between each pair of subspaces.

We use the isomorphism α to identify H with the group $G \rtimes_{s, \tau} Q$. With the identification, the homomorphism j maps the group $G \rtimes_{s, \tau} Q$ to Q by taking the second component. In particular, if an element f is in the component $Z(\mathbb{C}H)_{\langle q \rangle}$, f must be of the form

$$\sum_{g, q_0 \in \langle q \rangle} f_{g, q_0}(g, q_0).$$

Applying χ to f , we get an element

$$\chi(f) = \sum_{[\rho] \in \widehat{G}, q_0 \in \langle q \rangle} \sum_g f_{g, q_0}(\rho(g), q_0).$$

Since f is in the center of $\mathbb{C}H$, we can apply the map Φ to obtain an element of $Z(C(\widehat{G} \rtimes Q, c))$,

$$\begin{aligned} \Phi(\chi(f)) &= \sum_{\rho_1, i} \frac{1}{\dim(V_{\rho_1})} \Theta((\eta_{\rho_1}^i, 1)\chi(f), (\xi_i^{\rho_1}, 1)) \\ &= \sum_{q_0 \in \langle q \rangle} \sum_{[\rho_0] \in \widehat{G}} \sum_{g \in G} f_{g, q_0} \sum_{\rho_1, i} \frac{1}{\dim(V_{\rho_1})} \Theta((\eta_{\rho_1}^i, 1)(\rho_0(g), q_0), (\xi_i^{\rho_1}, 1)) \\ &= \sum_{q_0 \in \langle q \rangle} \sum_{[\rho_0] \in \widehat{G}, i} \sum_{g \in G} \frac{f_{g, q_0}}{\dim(V_{\rho_0})} \Theta((c^{[\rho_0]}(q_0, q_0^{-1})\eta_{\rho_0}^i \circ \rho_0(g) \circ T_{q_0}^{[\rho_0]-1}, q_0^{-1}), (\xi_i^{\rho_0}, 1)) \\ &= \sum_{q_0 \in \langle q \rangle} \sum_{\substack{[\rho_0] \in \widehat{G}, \\ q_0([\rho_0]) = [\rho_0]}} \sum_{i, g \in G} \frac{f_{g, q_0}}{\dim(V_{\rho_0})} (c^{[\rho_0]}(q_0, q_0^{-1})\eta_{\rho_0}^i \circ \rho_0(g) \circ T_{q_0}^{[\rho_0]-1}(\xi_i^{\rho_0})[\rho_0], q_0) \\ &= \sum_{q_0 \in \langle q \rangle} \sum_{\substack{[\rho_0] \in \widehat{G}, \\ q_0([\rho_0]) = [\rho_0]}} \sum_{i, g \in G} \frac{f_{g, q_0}}{\dim(V_{\rho_0})} (\eta_{\rho_0}^i \circ \rho_0(g) \circ T_{q_0}^{[\rho_0]-1}(\xi_i^{\rho_0})[\rho_0], q_0). \end{aligned}$$

From this computation, we see that $\Phi(\chi(f))$ belongs to $Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$. \square

By Theorem 3.3, $\mathbb{C}H$ is Morita equivalent to $C(\widehat{G} \rtimes Q, c)$. By Theorem 3.3, $C(\widehat{G} \rtimes Q, c)$ is Morita equivalent to $\bigoplus_{O_k \in \text{Orb } Q(\widehat{G})} C(\text{Stab}([\rho_k]), c_{[\rho_k]})$, where $[\rho_k]$ is an element in the orbit O_k . By the discussion in this section, with the Morita equivalence bimodules constructed in Theorems 3.3, we have an algebra isomorphism I ,

$$(3.10) \quad I : Z(\mathbb{C}H) \longrightarrow Z(\bigoplus_{O_k \in \text{Orb } Q(\widehat{G})} C(\text{Stab}([\rho_k]), c_{[\rho_k]})) = \bigoplus_{O_k \in \text{Orb } Q(\widehat{G})} Z(C(\text{Stab}([\rho_k]), c_{[\rho_k]})).$$

It is easy to check that on $\mathbb{C}H$ (respectively, on each $C(\text{Stab}([\rho_k]), c_{[\rho_k]})$), there is a canonical trace tr_H (respectively, $\text{tr}_{[\rho_k]}$) which takes the value⁹ $1/|H|$ (respectively, $1/|\text{Stab}([\rho_k])|$) on the identity element and 0 otherwise. Furthermore, the linear combination $\sum_k a_k \text{tr}_{[\rho_k]}$ for arbitrary coefficients a_k defines a trace on the algebra $\oplus_{O_k \in \text{Orb}^Q(\widehat{G})} C(\text{Stab}([\rho_k]), c_{[\rho_k]})$. tr_H restricts to define a trace on the center $Z(\mathbb{C}H)$, and $\sum_k a_k \text{tr}_{[\rho_k]}$ restricts to define a trace on the center $Z(\oplus_{O_k \in \text{Orb}^Q(\widehat{G})} C(\text{Stab}([\rho_k]), c_{[\rho_k]}))$. The next result identifies which trace on $Z(\oplus_{O_k \in \text{Orb}^Q(\widehat{G})} C(\text{Stab}([\rho_k]), c_{[\rho_k]}))$ pulls back to tr_H via I .

Proposition 3.5. *The isomorphism I pulls back the trace $\sum_k \frac{\dim(V_{\rho_k})^2}{|G|^2} \text{tr}_{[\rho_k]}$ to the trace tr_H .*

Proof. In the proof of Proposition 3.4, we obtained an explicit map I from $Z(\mathbb{C}H)$ to $Z(C(\widehat{G} \rtimes Q, c))$. We recall the formula of this map. If $f = \sum_{g,q} f_{g,q}(g, q)$ is in the center of $\mathbb{C}H$, then

$$\begin{aligned} I(f) &= \sum_{g,q} \sum_{[\rho] \in \widehat{G}, q([\rho])=[\rho], i} \frac{1}{\dim(V_\rho)} f_{g,q}(\eta_\rho^i \circ \rho(g) \circ T_q^{[\rho]-1}(\xi_i^\rho)[\rho], q) \\ &= \sum_{[\rho] \in \widehat{G}, q([\rho])=[\rho]} \sum_{i,g,q} \frac{1}{\dim(V_\rho)} f_{g,q}(\eta_\rho^i \circ \rho(g) \circ T_q^{[\rho]-1}(\xi_i^\rho)[\rho], q) \\ &= \sum_{O_k \in \text{Orb}^Q(\widehat{G})} \sum_{\substack{[\rho] \in O_k, \\ q([\rho])=[\rho]}} \sum_{i,g,q} \frac{1}{\dim(V_\rho)} f_{g,q}(\eta_\rho^i \circ \rho(g) \circ T_q^{[\rho]-1}(\xi_i^\rho)[\rho], q). \end{aligned}$$

Define

$$I(f)_k = \sum_{\substack{[\rho] \in O_k, \\ q([\rho])=[\rho]}} \sum_{i,g} \frac{1}{\dim(V_\rho)} f_{g,q}(\eta_\rho^i \circ \rho(g) \circ T_q^{[\rho]-1}(\xi_i^\rho)[\rho], q) \in C(O_k \rtimes Q, c).$$

Then $I(f) = \sum_{O_k \in \text{Orb}^Q(\widehat{G})} I(f)_k$, and $I(f)_k$ is an element in the center of $C(O_k \rtimes Q, c_k)$.

Using the data of Morita equivalent bimodules constructed in Theorem 3.3, we write down an explicit map I'_k from $Z(C(O_k \rtimes Q, c_k))$ to $Z(C(\text{Stab}([\rho_k]), c_{[\rho_k]}))$. If f_k is in $Z(C(O_k \rtimes Q, c_k))$, then

$$I'_k(f_k) := \sum_{q \in Q} \frac{c_{[\rho_k]}(q, q^{-1})^{-1}}{|Q|} X(\delta_{([\rho_k], q)} f_k, \delta_{(q([\rho_k]), q^{-1})}).$$

⁹We have chosen this particular normalization to make it compatible with the Poincaré pairing on BH .

Here the map X is defined in the proof of Theorem 3.3. Applying the above formula to the explicit expression of f_k , we have

$$\begin{aligned}
I'_k(f_k) &= \sum_{q \in Q} \frac{c^{[\rho_k]}(q, q^{-1})^{-1}}{|Q|} \sum_{\substack{[\rho_1] \in O_k, \\ q_1([\rho_1]) = [\rho_1]}} \sum_{i, g} \\
&\quad X\left(\delta_{([\rho_k], q)} \frac{1}{\dim(V_{\rho_1})} f_{g, q_1}(\eta_{\rho_1}^i \circ \rho_1(g) \circ T_{q_1}^{[\rho_1]-1}(\xi_i^{\rho_1})[\rho_1], q_1), \delta_{(q([\rho_k]), q^{-1})}\right) \\
&= \sum_{q \in Q} \frac{c^{[\rho_k]}(q, q^{-1})^{-1}}{|Q|} \sum_{\substack{[\rho_1] \in O_k, \\ q_1([\rho_1]) = [\rho_1]}} \sum_{i, g} \frac{1}{\dim(V_{\rho_1})} f_{g, q_1} \eta_{\rho_1}^i \circ \rho_1(g) \circ T_{q_1}^{[\rho_1]-1}(\xi_i^{\rho_1}) \\
&\quad X\left(\delta_{([\rho_k], q)}([\rho_1], q_1), \delta_{(q([\rho_k]), q^{-1})}\right) \\
&= \sum_{q \in Q} \sum_{q_1(q([\rho_k])) = q([\rho_k])} \sum_{i, g} \frac{c^{[\rho_k]}(q, q^{-1})^{-1}}{|Q| \dim(V_{q([\rho_k])})} f_{g, q_1} \eta_{q([\rho_k])}^i \circ q(\rho_k)(g) \circ T_{q_1}^{q([\rho_k])-1}(\xi_i^{q([\rho_k])}) \\
&\quad X\left(\delta_{([\rho_k], q)}(q([\rho_k]), q_1), \delta_{(q([\rho_k]), q^{-1})}\right) \\
&= \sum_{q \in Q} \sum_{q_1(q([\rho_k])) = q([\rho_k])} \sum_{i, g} \frac{c^{[\rho_k]}(q, q^{-1})^{-1}}{|Q| \dim(V_{q([\rho_k])})} f_{g, q_1} \eta_{q([\rho_k])}^i \circ q(\rho_k)(g) \circ T_{q_1}^{q([\rho_k])-1}(\xi_i^{q([\rho_k])}) \\
&\quad c^{[\rho_k]}(q, q_1) X\left(\delta_{([\rho_k], qq_1)}, \delta_{(q([\rho_k]), q^{-1})}\right) \\
&= \sum_{q \in Q} \sum_{q_1(q([\rho_k])) = q([\rho_k])} \sum_{i, g} \frac{c^{[\rho_k]}(q, q^{-1})^{-1}}{|Q| \dim(V_{q([\rho_k])})} f_{g, q_1} \eta_{q([\rho_k])}^i \circ q([\rho_k])(g) \circ T_{q_1}^{q([\rho_k])-1}(\xi_i^{q([\rho_k])}) \\
&\quad c^{[\rho_k]}(q, q_1) c^{[\rho_k]}(qq_1, q^{-1}) qq_1 q^{-1}.
\end{aligned}$$

We notice that if the coefficient of f_k on every component $([\rho], 1)$ for any $[\rho] \in O$ is equal to zero, then the coefficient of $I'_k(f_k)$ at the identity element of $C(\text{Stab}([\rho_k]), c_{[\rho_k]})$ is equal to zero. In particular, the evaluation of tr_{ρ_k} on $I'_k(f_k)$ is equal to zero. Hence

$$\begin{aligned}
\text{tr}_{\rho_k}(I'_k(f_k)) &= \frac{1}{|\text{Stab}([\rho_k])|} \sum_{q \in Q} \sum_{i, g} \frac{1}{|Q| \dim(V_{q([\rho_k])})} f_{g, 1} \eta_{q([\rho_k])}^i \circ q([\rho_k])(g) (\xi_i^{q([\rho_k])}) \\
&= \sum_{\rho \in O_k} \sum_{i, g} \frac{1}{|Q| \dim(V_{\rho})} f_{g, 1} \eta_{\rho}^i \circ \rho(g) (\xi_i^{\rho}),
\end{aligned}$$

where in the last equality, we used the fact that for $q \in \text{Stab}([\rho_k])$, $q([\rho_k]) = [\rho_k]$.

Now summing over all orbits $O_k \in \text{Orb}^Q(\hat{G})$, we have that

$$\begin{aligned}
\sum_{O_k \in \text{Orb}^Q(\hat{G})} \frac{\dim(V_{\rho_k})^2}{|G|} \text{tr}_{\rho_k}(I'(f)) &= \sum_{\rho \in \hat{G}} \sum_{i, g} \frac{\dim(V_{\rho})^2}{|G|^2 |Q| \dim(V_{\rho})} f_{g, 1} \eta_{\rho}^i \circ \rho(g) (\xi_i^{\rho_k}) \\
&= \sum_g f_{g, 1} \frac{1}{|H|} \sum_{\rho, i} \frac{\dim(V_{\rho})}{|G|} \eta_{\rho}^i \circ \rho(g) (\xi_i^{\rho_k}) \\
&= \sum_g f_{g, 1} \frac{1}{|H|} \sum_{\rho} \dim(V_{\rho}) \text{tr}_{\rho}(g),
\end{aligned}$$

where we used the fact that $\dim(V_\rho) = \dim(V_{\rho_k})$ if $[\rho] \in O_k$, and tr_ρ is the $1/|G|$ times the standard trace on $\text{End}(V_\rho)$. A standard result (see, e.g., [32, Exercise 3.32]) in group representation theory implies that

$$\sum_g f_{g,1} \frac{1}{|H|} \sum_\rho \dim(V_\rho) \text{tr}_\rho(g) = \sum_g f_{g,1} \frac{1}{|H|} \text{tr}_G(g) = \frac{1}{|H|} f_{1,1} = \text{tr}_H(f).$$

This proves that the pull back of $\sum_k \frac{\dim(V_{\rho_k})^2}{|G|^2} \text{tr}_{[\rho_k]}$ along the algebra isomorphism I is equal to tr_H . \square

Remark 3.4. *The explicit formula for the isomorphism I ,*

$$I(f) = \sum_q \sum_{\substack{[\rho_0] \in \widehat{G} \\ q([\rho]) = [\rho]}} \sum_g \frac{f_{g,q}}{\dim(V_\rho)} \text{tr}(\rho(g) T_q^{[\rho]-1})([\rho], q),$$

is crucial in the above proofs of Propositions 3.4-3.5. In Section 4.3, we will give a generalization of the formula for I on a G -gerbe. See equation (4.15).

4. GROUPOID ALGEBRAS AND HOCHSCHILD COHOMOLOGY OF GERBES ON ORBIFOLDS

In this section, we generalize the formulation of the Mackey machine in Section 3.2 to groupoids. Our results relate several aspects of the geometry of a G -gerbe \mathcal{Y} on an orbifold \mathcal{B} to aspects of the geometry of the dual orbifold $\widehat{\mathcal{Y}}$ twisted by the flat $U(1)$ -gerbe c .

4.1. Global quotient. In this subsection, we consider the special case of the global quotient, which is very close to what we developed in Section 3.2. Consider the group extension as in equation (3.1)

$$1 \rightarrow G \rightarrow H \rightarrow Q \rightarrow 1.$$

Let M be a smooth manifold. We assume that H acts on M by diffeomorphisms so that G is contained in the kernel of the group homomorphism $H \rightarrow \text{Diff}(M)$, i.e., the induced G -action on M is trivial. Therefore, the composition of $H \rightarrow \text{Diff}(M)$ with a section $s : Q \rightarrow H$ of (3.1) yields a well-defined Q -action on M . Consequently, this defines a G -gerbe

$$(4.1) \quad [M/H] \rightarrow [M/Q].$$

In terms of a transformation groupoid, this gives the following extension of Lie groupoids, which generalizes equation (3.1),

$$(4.2) \quad 1 \rightarrow M \times G \rightarrow M \rtimes H \rightarrow M \rtimes Q \rightarrow 1.$$

We follow the idea in Section 3.2 to study the above groupoid extension. First, note that the section s for the extension (3.1) defines a section of the groupoid extension (4.2).

Let \mathcal{A} be a sheaf of algebras on M with an H action. In this paper, \mathcal{A} is either the sheaf of smooth functions on M or the sheaf of deformation quantization of M when M is a symplectic manifold. Such a sheaf \mathcal{A} with an H action is called a \mathfrak{H} -sheaf on the transformation groupoid $\mathfrak{H} := M \rtimes H \rightrightarrows M$. We are interested in the crossed product algebra $\mathcal{A} \rtimes H$.

Similar to the approach in Section 3.2, we use the data about G and Q to study the algebra $\mathcal{A} \rtimes H$. With the same notations, we consider the space $M \times \widehat{G}$ equipped with the diagonal Q -action. Here we use the Q -action on \widehat{G} as defined in Section 3.1. The dual orbifold associated to the G -gerbe (4.1) is the quotient $\widehat{\mathcal{Y}} = [M \times \widehat{G}/Q]$. On $M \times \widehat{G}$, there is a vector bundle¹⁰

$$\mathcal{V}_G \rightarrow M \times \widehat{G}.$$

¹⁰Here we allow the ranks of a vector bundle to be different on different connected components.

For every $[\rho] \in \widehat{G}$, the bundle \mathcal{V}_G on the component $M \times [\rho]$ is equal to $M \times V_\rho \rightarrow M \times [\rho]$. It is important to observe that the vector bundle \mathcal{V}_G is not Q -equivariant. Instead, the cocycle c defined in Proposition 3.1 determines the obstruction for \mathcal{V}_G to be Q -equivariant. More precisely, \mathcal{V}_G is a c^{-1} -twisted Q -equivariant vector bundle on $M \times \widehat{G}$. See Lemma 7.1 below. In the language of gerbes, this means that there is a $U(1)$ -gerbe over the orbifold $[M \times \widehat{G}/Q]$ on which \mathcal{V}_G becomes a vector bundle. Furthermore, as Q is finite, this $U(1)$ -gerbe is flat and it may be represented by the cocycle c^{-1} .

The H -equivariant sheaf \mathcal{A} on M lifts to define a Q -equivariant sheaf $\tilde{\mathcal{A}}$ on $M \times \widehat{G}$. Using the $U(1)$ -valued 2-cocycle c , we define a twisted crossed product algebra $\tilde{\mathcal{A}} \rtimes_c Q$ to be the space $\Gamma(\tilde{\mathcal{A}})$ of sections of $\tilde{\mathcal{A}}$ with the following product:

$$(a_1, q_1) \circ (a_2, q_2) := (a_1 q_1(a_2) c(q_1, q_2), q_1 q_2), \quad a_1, a_2 \in \Gamma(\tilde{\mathcal{A}}),$$

where $c(q_1, q_2)$ is an element in $U(1)$ identified as a scalar on the unit circle of \mathbb{C} .

Proposition 4.1. *The crossed product algebra $\mathcal{A} \rtimes H$ is Morita equivalent to the twisted crossed product algebra $\tilde{\mathcal{A}} \rtimes_c Q$.*

Proof. The proof is along the line of the proof of Theorem 3.3. We will define the bimodules, and omit the details of checking the equivalence bimodule structures. Set $A := \mathcal{A} \rtimes H$ and $B := \tilde{\mathcal{A}} \rtimes_c Q$.

Consider the vector bundle \mathcal{V}_G over $M \times \widehat{G}$. Define an A - B bimodule \mathcal{M} by

$$\mathcal{M} := \Gamma(\tilde{\mathcal{A}} \otimes \mathcal{V}_G) \times Q.$$

The left A -module structure on \mathcal{M} is defined by

$$(a_0, x_{\rho_0}, q_0)(a, \xi_\rho, q) := \begin{cases} (a_0 q_0(a), c^{[\rho_0]}(q_0, q) x_{\rho_0} T_{q_0}^{[\rho_0]-1}(\xi_{q_0([\rho_0])}), q_0 q) & \text{if } [\rho] = q_0([\rho_0]) \\ 0 & \text{otherwise} \end{cases}.$$

The right B -module structure on \mathcal{M} is defined by

$$(a, \xi_\rho, q)(a_1, [\rho_1], q_1) := \begin{cases} (aq(a_1), c^{[\rho]}(q, q_1) \xi_\rho, qq_1) & \text{if } [\rho_1] = q([\rho]) \\ 0 & \text{otherwise} \end{cases}.$$

Consider the dual bundle \mathcal{V}_G^* over $M \times \widehat{G}$. Define a B - A bimodule \mathcal{N} by

$$\mathcal{N} := \Gamma(\tilde{\mathcal{A}} \otimes \mathcal{V}_G^*) \times Q.$$

The right A -module structure on \mathcal{N} is defined by

$$(a, \eta_\rho, q)(a_0, x_{\rho_0}, q_0) = \begin{cases} (q_0^{-1}(aa_0), c^{[\rho_0]}(q_0, q_0^{-1}q) \eta_{\rho_0} \circ x_{\rho_0} \circ T_{q_0}^{[\rho_0]-1}, q_0^{-1}q) & \text{if } [\rho] = [\rho_0] \\ 0 & \text{otherwise} \end{cases},$$

and the left B -module structure on \mathcal{N} is defined by

$$(a_1, [\rho_1], q_1)(a, \eta_\rho, q) := \begin{cases} (qq_1^{-1}(a_1)a, c^{[\rho]}(qq_1^{-1}, q_1) \eta_\rho, qq_1^{-1}) & \text{if } [\rho_1] = qq_1^{-1}([\rho]) \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define an isomorphism $\Theta : \mathcal{N} \otimes_A \mathcal{M} \rightarrow B$ of B - B bimodules by

$$\Theta((a, \eta_\rho, q), (a', \xi_{\rho'}, q')) = \begin{cases} (q^{-1}(aa'), \eta_\rho(\xi_{\rho'}) \frac{1}{c^{[\rho]}(q, q^{-1}q')} q([\rho]), q^{-1}q') & \text{if } [\rho'] = [\rho] \\ 0 & \text{otherwise,} \end{cases}$$

and an isomorphism $\Xi : \mathcal{M} \otimes_B \mathcal{N} \rightarrow A$ of A - A bimodules by

$$\begin{aligned} & \Xi((a, \xi_\rho, q), (a', \eta_{\rho'}, q')) \\ & := \begin{cases} (aq q'^{-1}(a'), \xi_\rho \otimes \eta_{qq'^{-1}([\rho])} \circ T_{qq'^{-1}}^{[\rho]} \frac{c^{qq'^{-1}([\rho])}(q'q^{-1}, q)}{c^{qq'^{-1}([\rho])}(q'q^{-1}, qq'^{-1})}, qq'^{-1}) & \text{if } qq'^{-1}([\rho]) = [\rho'] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

Remark 4.2. As established in Lemma 7.1 below, the sheaf \mathcal{V}_G is a twisted sheaf over $[M \times \widehat{G}/Q]$. The bimodules \mathcal{M} and \mathcal{N} introduced in the above proof of Proposition 4.1 can be “thought of” as the space of sections of the twisted sheaves $\widetilde{\mathcal{A}} \otimes \mathcal{V}_G$ and $\widetilde{\mathcal{A}} \otimes \mathcal{V}_G^*$ over the orbifold $[M \times \widehat{G}/Q]$.

Similar to Proposition 4.1, we can easily generalize Theorem 3.3.

Proposition 4.3. The algebra $\widetilde{\mathcal{A}} \rtimes_c Q$ is Morita equivalent to the algebra $\bigoplus_{O \in \text{Orb}^Q(\widehat{G})} \mathcal{A} \rtimes_{c_{\rho_O}} \text{Stab}(\rho_O)$, where ρ_O is the chosen representative of an element $[\rho_O] \in O$ in the orbit O , and c_{ρ_O} is the cocycle obtained from c by restriction.

Proof. The proof closely follows the proof of Theorem 3.3 and is left to the reader. □

In summary, we have proved the following:

Theorem 4.4. The crossed product algebra $\mathcal{A} \rtimes H$ is Morita equivalent to the direct sum

$$\bigoplus_{O \in \text{Orb}^Q(\widehat{G})} \mathcal{A} \rtimes_{c_{\rho_O}} \text{Stab}(\rho_O)$$

of twisted crossed product algebras.

We remark that the above theorem may be viewed as a natural generalization of the Mackey machine to the groupoid $M \rtimes H$.

4.2. General case. In this subsection, we study a generalization of the groupoid extension equation (4.2). Let $\mathfrak{H}, \mathfrak{Q}, \mathfrak{G}$ be proper étale groupoids and $i : \mathfrak{G} \rightarrow \mathfrak{H}$ and $j : \mathfrak{H} \rightarrow \mathfrak{Q}$ be groupoid morphisms that fit in the exact sequence

$$(4.3) \quad \mathfrak{G} \xrightarrow{i} \mathfrak{H} \xrightarrow{j} \mathfrak{Q}.$$

We make the following assumptions:

- (1) The groupoid \mathfrak{G} is a principal G bundle over its unit space \mathfrak{G}_0 .
- (2) The morphisms i and j are identity maps when restricted to the unit space.

According to [44], the Morita equivalence class of the above G -groupoid extension defines a G -gerbe $\mathcal{Y} = [\mathfrak{H}_0/\mathfrak{H}]$ on the orbifold $\mathcal{B} = [\mathfrak{Q}_0/\mathfrak{Q}]$ associated with the groupoid \mathfrak{Q} , and every G -gerbe over \mathcal{B} arises in this way. We are interested in the geometry of the G -gerbe $\mathcal{Y} \rightarrow \mathcal{B}$.

We recall the procedure of refining an étale groupoid $\mathcal{G} \rightrightarrows \mathcal{G}_0$ by a countable open covering \mathcal{U} of \mathcal{G}_0 . For every $U_i, U_j \in \mathcal{U}$, consider the subspace $\mathcal{G}_{U_j}^{U_i} := \{g \in \mathcal{G}, s(g) \in U_i, t(g) \in U_j\}$. Define a new groupoid $\mathcal{G}_{\mathcal{U}}$ by $\coprod_{i,j} \mathcal{G}_{U_j}^{U_i} \rightrightarrows \coprod_i U_i$. Moerdijk and Pronk [51] proved that $\mathcal{G}_{\mathcal{U}}$ is Morita equivalent to \mathcal{G} , and that if we choose every U_i to be sufficiently small, the groupoid $\mathcal{G}_{\mathcal{U}}$ and every level of its nerve space are disjoint unions of contractible open sets. (Let \mathcal{G} be a groupoid. For $n \geq 0$, the n -th level \mathcal{G}_n of the nerve space of \mathcal{G} is defined to be the space of n composable arrows in \mathcal{G} .)

Laurent-Gengoux, Stiénon, and Xu proved [44] that a groupoid extension like equation (4.3) always has a refinement with respect to a covering \mathcal{U} of \mathfrak{Q}_0 such that the kernel of the groupoid extension is a G trivial bundle. Combining this result with the result from Moerdijk and Pronk [51], we conclude that there is a covering \mathcal{U} of \mathfrak{Q}_0 such that the associated refinement of the groupoid extension

$$\coprod_i U_i \times G \rightarrow \mathfrak{H}_{\mathcal{U}} \rightarrow \mathfrak{Q}_{\mathcal{U}}$$

is Morita equivalent to the original groupoid extension (4.3) and $\mathfrak{Q}_{\mathcal{U}}$ and every level of its nerve space are disjoint unions of contractible open sets. Since $\mathfrak{H}_{\mathcal{U}}$ is Morita equivalent to \mathfrak{H} , the groupoid algebra

of $\mathfrak{H}_{\mathcal{U}}$ is Morita equivalent to the groupoid algebra of \mathfrak{H} . For the purpose of studying the geometry of the gerbe \mathcal{Y} we may thus study the groupoid algebra of $\mathfrak{H}_{\mathcal{U}}$.

To simplify notations, in what follows, we will always work with the groupoid extension

$$M \times G \xrightarrow{i} \mathfrak{H} \xrightarrow{j} \mathfrak{Q},$$

where $\mathfrak{H}_0 = \mathfrak{Q}_0 = M$, and \mathfrak{Q} is a disjoint union of contractible open sets. We generalize the Mackey machine formulated in Section 3.2 to study the above G -extension of proper étale groupoids.

As \mathfrak{Q} is a disjoint union of contractible open sets, the principal G bundle \mathfrak{H} over \mathfrak{Q} has a global section σ , i.e., there is a smooth map

$$\sigma : \mathfrak{Q} \rightarrow \mathfrak{H}$$

such that $j \circ \sigma = id$, and the restriction of σ to the unit space \mathfrak{Q}_0 is the identity map. Fix such a choice of σ .

Lemma 4.5. *For a section $\sigma : \mathfrak{Q} \rightarrow \mathfrak{H}$, $s(\sigma(q)) = s(q)$ and $t(\sigma(q)) = t(q)$ for any $q \in \mathfrak{Q}$.*

Proof. As $j \circ \sigma = id$, we have $s(q) = s(j \circ \sigma(q)) = j \circ s(\sigma(q)) = s(\sigma(q))$. The same argument works for the target map t . \square

We study the failure of σ to be a groupoid morphism. If σ is a groupoid morphism, then the above extension is a semi-direct product of the groupoid \mathfrak{Q} and the bundle $M \times G$. In general, for two composable arrows $q_1, q_2 \in \mathfrak{Q}$, Lemma 4.5 implies that $\sigma(q_1)$ and $\sigma(q_2)$ are composable arrows in \mathfrak{H} . However, $\sigma(q_1)\sigma(q_2)$ usually differs from $\sigma(q_1q_2)$. Generalizing (3.2), we define a map

$$\tau : \mathfrak{Q} \times_{\mathfrak{Q}_0} \mathfrak{Q} \rightarrow \mathfrak{H}^{(0)}, \quad \tau(q_1, q_2) := \sigma(q_1)\sigma(q_2)\sigma(q_1q_2)^{-1},$$

where $\mathfrak{H}^{(0)}$ is the loop space in \mathfrak{H} consisting of arrows $h \in \mathfrak{H}$ such that $s(h) = t(h)$. By Lemma 4.5, $\tau(q_1, q_2)$ is a well-defined element in \mathfrak{H} such that $s(\tau(q_1, q_2)) = s(\sigma(q_1)) = s(q_1) = t(\sigma(q_1q_2)^{-1}) = t(\tau(q_1, q_2))$. We also observe that

$$j(\tau(q_1, q_2)) = j(\sigma(q_1)\sigma(q_2)\sigma(q_1q_2)^{-1}) = j(\sigma(q_1))j(\sigma(q_2))j(\sigma(q_1q_2)^{-1}) = s(q_1).$$

This shows that $\tau(q_1, q_2)$ is in the kernel of j . Hence, we can consider $\tau(q_1, q_2)$ as an element in G .

We compute the cocycle condition that τ satisfies. For composable arrows $q_1, q_2, q_3 \in \mathfrak{Q}$,

$$\begin{aligned} & \tau(q_1, q_2)\tau(q_1q_2, q_3)\sigma(q_1q_2q_3) \\ &= (\sigma(q_1)\sigma(q_2))\sigma(q_3) \\ &= \sigma(q_1)(\sigma(q_2)\sigma(q_3)) \\ &= \sigma(q_1)\tau(q_2, q_3)\sigma(q_2q_3) \\ &= \sigma(q_1)\tau(q_2, q_3)\sigma(q_1)^{-1}\tau(q_1, q_2q_3)\sigma(q_1q_2q_3). \end{aligned}$$

Note that $s(\tau(q_2, q_3)) = t(\tau(q_2, q_3)) = s(q_2) = t(q_1)$. Hence,

$$\text{Ad}_{\sigma(q_1)}(\tau(q_2, q_3)) := \sigma(q_1)\tau(q_2, q_3)\sigma(q_1)^{-1}$$

is well-defined. In summary, we have the following equation, which generalizes (3.3):

$$(4.4) \quad \tau(q_1, q_2)\tau(q_1q_2, q_3) = \text{Ad}_{\sigma(q_1)}(\tau(q_2, q_3))\tau(q_1, q_2q_3).$$

With the above preparation, we have a new description of the groupoid \mathfrak{H} . Define $G \rtimes_{\sigma, \tau} \mathfrak{Q} \rightrightarrows \mathfrak{Q}_0$ to be the groupoid with the following structures. The space of arrows of this groupoid is $G \times \mathfrak{Q}$. Given an arrow $(g, q) \in G \times \mathfrak{Q}$, the source map takes (g, q) to $s(q)$ and the target map takes (g, q) to $t(q)$. Given composable arrows (g_1, q_1) and (g_2, q_2) , their product is defined to be

$$(g_1, q_1)(g_2, q_2) = (g_1 \text{Ad}_{\sigma(q_1)}(g_2)\tau(q_1, q_2), q_1q_2).$$

Define the isomorphism $\alpha : \mathfrak{H} \rightarrow G \rtimes_{\sigma, \tau} \mathfrak{Q}$ by

$$\alpha(h) = (h\sigma(j(h))^{-1}, j(h)).$$

The same argument that proves equation (3.4) now shows that α is a groupoid isomorphism. In the following construction, we will always work with the groupoid $G \rtimes_{\sigma, \tau} \mathfrak{Q}$.

Let \mathcal{A} be a \mathfrak{Q} -sheaf of algebras over \mathfrak{Q}_0 . Pulling back \mathcal{A} along the groupoid morphism j , we obtain an $\mathfrak{H} = G \rtimes_{\sigma, \tau} \mathfrak{Q}$ -sheaf¹¹ \mathcal{A} of algebras over $\mathfrak{H}_0 = \mathfrak{Q}_0$ such that every element in the kernel of j acts on \mathcal{A} trivially. We are interested in the crossed product algebra $\mathcal{A} \rtimes \mathfrak{H}$, which via the isomorphism α , is isomorphic to the crossed product algebra $\mathcal{A} \rtimes (G \rtimes_{\sigma, \tau} \mathfrak{Q})$. As the G component acts on \mathcal{A} trivially, the crossed product algebra $\mathcal{A} \rtimes (G \rtimes_{\sigma, \tau} \mathfrak{Q})$ is isomorphic to $(\mathcal{A} \otimes \mathbb{C}G) \rtimes_{\sigma, \tau} \mathfrak{Q}$.

Recall that in Section 3.2, for any class $[\rho] \in \widehat{G}$ we fix a choice of an irreducible representation in the class $[\rho]$ denoted by $\rho : G \rightarrow \text{End}(V_\rho)$. Also, there is an isomorphism $\mathbb{C}G \cong \bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho)$ of algebras. We replace $\mathbb{C}G$ by $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho)$ and study the \mathfrak{Q} action on the sheaf $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho)$.

Let q be an element in \mathfrak{Q} . The adjoint action $\text{Ad}_{\sigma(q)}$ defines a group homomorphism on G . Accordingly, $\text{Ad}_{\sigma(q)}$ acts on \widehat{G} as follows: for $[\rho] \in \widehat{G}$, the class $q([\rho]) \in \widehat{G}$ is the class of the G representation defined by $g \mapsto \rho(\text{Ad}_{\sigma(q)}(g))$. Again, by abuse of notation, the chosen irreducible G -representation that represents the class $q([\rho])$ will also be denoted by $q([\rho]) : G \rightarrow \text{End}(V_{q([\rho])})$. Since the representation $q([\rho]) : G \rightarrow \text{End}(V_{q([\rho])})$ is, by definition, equivalent to the representation $G \rightarrow \text{End}(V_\rho)$ defined by $g \mapsto \rho(\text{Ad}_{\sigma(q)}(g))$, there is an intertwining isomorphism $T_q^{[\rho]} : V_\rho \rightarrow V_{q([\rho])}$ such that $\rho(\text{Ad}_{\sigma(q)}(g)) = T_q^{[\rho]-1} \circ q([\rho])(g) \circ T_q^{[\rho]}$. For a pair of composable arrows $q_1, q_2 \in \mathfrak{Q}$, we have the following equation generalizing (3.5):

$$T_{q_2}^{q_1([\rho])} \circ T_{q_1}^{[\rho]} = c^{[\rho]}(q_1, q_2) T_{q_1 q_2}^{[\rho]} \rho(\tau(q_1, q_2))^{-1}.$$

The action¹² of \mathfrak{Q} on \widehat{G} defines a transformation groupoid $\widehat{G} \rtimes \mathfrak{Q}$. Since we have assumed that every level of the nerve space of \mathfrak{Q} is a disjoint union of contractible open sets, every level of the nerve space of the groupoid $\widehat{G} \rtimes \mathfrak{Q}$ is also a disjoint union of contractible open sets. Generalizing Proposition 3.1, we obtain the following result whose proof is left to the reader.

Proposition 4.6. *The $U(1)$ -valued function*

$$c : \widehat{G} \times \mathfrak{Q} \times_{\mathfrak{Q}_0} \mathfrak{Q} \rightarrow U(1), \quad ([\rho], q_1, q_2) \mapsto c^{[\rho]}(q_1, q_2)$$

defines a groupoid cocycle on $\widehat{G} \rtimes \mathfrak{Q}$ such that $c^\bullet(q, -) = c^\bullet(-, q) = 1$ if $q \in \mathfrak{Q}_0$.

Proposition 4.7. *The cocycle c introduced in Proposition 4.6 can be chosen to be locally constant. Consequently the corresponding gerbe c defined on the orbifold $\widehat{\mathcal{Y}} = [\widehat{G} \times \mathfrak{Q}_0 / \mathfrak{Q}]$ is flat.*

Proof. Note that given any $q \in \mathfrak{Q}$, $\text{Ad}_{\sigma(q)}(\cdot)$ defines a group automorphism of G . Hence, we have a smooth map $\text{Ad} : \mathfrak{Q} \rightarrow \text{Aut}(G)$. The set of group automorphism of G is a finite set. Therefore, the map $\text{Ad} : \mathfrak{Q} \rightarrow \text{Aut}(G)$ must be locally constant. Similarly, as G is finite, the continuous map $\tau : \mathfrak{Q} \times_{\mathfrak{Q}_0} \mathfrak{Q} \rightarrow G$ is also locally constant. Now following the construction in Section 3.2, we can associate to an automorphism ν of G a collection of isomorphisms $T_\nu^{[\rho]} : V_\rho \rightarrow V_{\nu([\rho])}$. Hence, we can choose $T_q^{[\rho]} : V_\rho \rightarrow V_{q([\rho])}$ to be locally constant in $q \in \mathfrak{Q}$. This implies that we can choose $c : \mathfrak{Q} \times_{\mathfrak{Q}_0} \mathfrak{Q} \rightarrow U(1)$ to be a locally constant cocycle. By [43, Proposition 3.26], we conclude that the $U(1)$ -gerbe defined by c over $\widehat{\mathcal{Y}}$ is flat. \square

¹¹Since $\mathfrak{Q}_0 = \mathfrak{H}_0$, we use \mathcal{A} to denote the same sheaf on \mathfrak{H}_0 but equipped with an \mathfrak{H} action.

¹²We will write this action as a left action.

With the above structure, the same arguments used in Proposition 3.2 prove that the crossed product algebra $(\mathcal{A} \otimes \mathbb{C}G) \rtimes_{\sigma, \tau} \mathfrak{Q}$ is isomorphic to the crossed product algebra $(\mathcal{A} \otimes \bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho)) \rtimes_{T, c} \mathfrak{Q}$, whose product structure is defined by

$$\begin{aligned} & ((a \otimes x) \circ (a' \otimes x'))(q) \\ &:= \sum_{q_1 q_2 = q, [\rho]} a(q_1) q_1(a'(q_2)) \otimes x(q_1)_{[\rho]} T_{q_1}^{[\rho](q_1)-1} x'(q_2)_{q_1([\rho])} T_{q_1}^{[\rho]} \rho(\tau(q_1, q_2)), \end{aligned}$$

where x is a section of the sheaf $s^*(\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho))$ over \mathfrak{Q} (a trivial bundle over \mathfrak{Q}), and we use $x(q)_{[\rho]}$ to denote the $[\rho]$ component of x at q .

Let \mathcal{A} be a \mathfrak{Q} -sheaf of algebras over \mathfrak{Q}_0 . This defines a $\widehat{G} \rtimes \mathfrak{Q}$ -sheaf $\widetilde{\mathcal{A}}$ of algebras on $\widehat{G} \times \mathfrak{Q}_0$. Using the $U(1)$ -valued groupoid 2-cocycle on $\widehat{G} \rtimes \mathfrak{Q}$, which was defined in Proposition 4.6, we define a c -twisted crossed product algebra $\widetilde{\mathcal{A}} \rtimes_c (\widehat{G} \rtimes \mathfrak{Q})$ by

$$(4.5) \quad a_1 \circ a_2([\rho], q) = \sum_{q=q_1 q_2} a_1([\rho], q_1) q_1 \left(a_2(q_1([\rho]), q_2) \right) c^{[\rho]}(q_1, q_2).$$

In the following developments, the property that c is a locally constant 2-cocycle plays a crucial role. Otherwise, the product defined in (4.5) is not even associative.

Theorem 4.8. *The crossed product algebra $\mathcal{A} \rtimes \mathfrak{H}$ is Morita equivalent to the twisted crossed product algebra $\widetilde{\mathcal{A}} \rtimes_c (\widehat{G} \rtimes \mathfrak{Q})$.*

Remark 4.9. *In general, the algebras $\mathcal{A} \rtimes \mathfrak{H}$ and $\widetilde{\mathcal{A}} \rtimes_c (\widehat{G} \rtimes \mathfrak{Q})$ are quasi-unital algebras [56, Appendix A]. We use the methods of [56, Appendix A] to work with Morita equivalence of quasi-unital algebras.*

Proof. The proof is a straightforward generalization of the proofs of Theorems 3.3 and 4.4. We will define the bimodules and omit the lengthy but straightforward computations, which is similar to those in the proof of Theorem 3.3. Let $A = \mathcal{A} \rtimes \mathfrak{H}$ and $B = \widetilde{\mathcal{A}} \rtimes_c (\widehat{G} \rtimes \mathfrak{Q})$.

Let s be the source map from \mathfrak{Q} to \mathfrak{Q}_0 . The pull-back $s^*(\mathcal{A} \otimes (\bigoplus_{[\rho] \in \widehat{G}} V_\rho))$ defines a sheaf over \mathfrak{Q} . Let $\mathcal{M} := \Gamma_{\text{cpt}}\left(s^*(\mathcal{A} \otimes (\bigoplus_{[\rho] \in \widehat{G}} V_\rho))\right)$, where $\Gamma_{\text{cpt}}(-)$ is the set of compactly supported sections. The left A -module structure on \mathcal{M} is defined by

$$\begin{aligned} & ((a \otimes x)(a_0 \otimes \xi))(q) \\ &= \sum_{q=q_1 q_2, [\rho]} a(q_1) q_1(a(q_2)) \otimes c^{[\rho]}(q_1, q_2) x(q_1)_{[\rho]} T_{q_1}^{[\rho]-1} (\xi(q_2)_{q_1([\rho])}); \end{aligned}$$

and the right B -module structure on \mathcal{M} is defined by

$$((a_0 \otimes \xi)(a_1))(q) := \sum_{q=q_1 q_2, [\rho]} a_0(q_1) q_1 \left(a_1(q_1([\rho]), q_2) \right) c^{[\rho]}(q_1, q_2) \otimes \xi(q_1)_{[\rho]}.$$

Consider the sheaf $\mathcal{A} \otimes (\bigoplus_{[\rho] \in \widehat{G}} V_\rho^*)$ on \mathfrak{Q}_0 . The pull-back $s^*(\mathcal{A} \otimes (\bigoplus_{[\rho] \in \widehat{G}} V_\rho^*))$ defines a sheaf over \mathfrak{Q} . Let $\mathcal{N} := \Gamma_{\text{cpt}}\left(s^*(\mathcal{A} \otimes (\bigoplus_{[\rho] \in \widehat{G}} V_\rho^*))\right)$ be a B - A bimodule. The left B -module structure on \mathcal{N} is defined by

$$((a_1)(a \otimes \eta))(q) := \sum_{q=q_2 q_1^{-1}, [\rho]} (q_2 q_1^{-1})(a_1(q_2 q_1^{-1}([\rho]), q_1)) a(q_2) c^{[\rho]}(q_2 q_1^{-1}, q_1) \otimes \eta(q_2)_{[\rho]};$$

and the right A -module structure on \mathcal{N} is defined by

$$\begin{aligned} & ((a \otimes \eta)(a_0 \otimes x))(q) \\ &= \sum_{q=q_2^{-1}q_1, [\rho]} q_2^{-1}(a(q_1)a_0(q_2))c^{q_2^{-1}([\rho])}(q_2, q_2^{-1}q_1) \otimes \eta(q_1)_{q_2^{-1}([\rho])} \circ x(q_2)_{q_2^{-1}([\rho])} \circ T_{q_2}^{q_2^{-1}([\rho])^{-1}}. \end{aligned}$$

We define an A - A bimodule map $\Xi : \mathcal{M} \otimes_B \mathcal{N} \rightarrow A$ by

$$\begin{aligned} & \Xi(a \otimes \xi, a' \otimes \eta)(q) \\ &:= \sum_{q=q_1q_2^{-1}, [\rho]} (a(q_1)q_1q_2^{-1}(a'(q_2)) \\ & \quad \otimes (\xi(q_1)_{[\rho]} \otimes \eta(q_2)_{q_1q_2^{-1}([\rho])} \circ T_{q_1q_2^{-1}}^{[\rho]} \frac{c^{q_1q_2^{-1}([\rho])}(q_2q_1^{-1}, q_1)}{c^{q_1q_2^{-1}([\rho])}(q_2q_1^{-1}, q_1q_2^{-1})})). \end{aligned} \quad (4.6)$$

We define an isomorphism $\Theta : \mathcal{N} \otimes_A \mathcal{M} \rightarrow B$ as a B - B bimodule,

$$\Theta((a \otimes \eta), (a' \otimes \xi))(\rho_0, q) = \sum_{q=q_1^{-1}q_2} \frac{(q_1^{-1}(a(q_1)a'(q_2))\eta(q_1)_{q_1^{-1}([\rho_0])}(\xi(q_2)_{q_1^{-1}([\rho_0])}))}{c^{q_1^{-1}([\rho])}(q_1, q_1^{-1}q_2)}.$$

□

Remark 4.10. The proof of Theorem 4.8 leaves no doubt that the corresponding C^* -algebra completions of $\mathcal{A} \rtimes \mathfrak{H}$ and $\tilde{\mathcal{A}} \rtimes_c (\hat{G} \rtimes \mathfrak{Q})$ are also Morita equivalent. Since we do not need this result in this paper, we will not discuss its proof in this paper.

Since Morita equivalent algebras have isomorphic categories of modules, we deduce the following corollary from [56, Theorem A.12].

Corollary 4.11. The algebras $\mathcal{A} \rtimes \mathfrak{H}$ and $\tilde{\mathcal{A}} \rtimes_c (\hat{G} \rtimes \mathfrak{Q})$ have isomorphic categories of modules. In particular, the K -theory and (Hochschild) cohomology groups of the two algebras are isomorphic.

Remark 4.12. As explained in [56, Theorem A.12], in order to have an isomorphism between the Hochschild cohomologies of Morita equivalent algebras, one needs the maps $\Xi : \mathcal{M} \otimes_B \mathcal{N} \rightarrow A$ and $\Theta : \mathcal{N} \otimes_A \mathcal{M} \rightarrow B$ to satisfy

$$q\Xi(p, q') = \Theta(q, p)q', \quad p\Theta(q, p') = \Xi(p, q)p'$$

for any p, p' in \mathcal{M} and q, q' in \mathcal{N} . We can easily check the above identities with the explicit formulas for Ξ and Θ from Remark 3.1.

4.3. Cohomology. In this subsection, we study geometric consequences derived from the Morita equivalence results, Theorem 4.8 and Corollary 4.11, in the previous subsection. Here, we always work with bornological algebras and bornological tensor products. All of the algebras considered in this paper are Fréchet algebras, and the bornologies on them are the ones associated with the Fréchet topologies.

We start with the K -theory of the algebras $\mathcal{A} \rtimes \mathfrak{H}$ and $\tilde{\mathcal{A}} \rtimes_c (\hat{G} \rtimes \mathfrak{Q})$. For this purpose, we will consider \mathcal{A} to be the sheaf \mathcal{C}^∞ of smooth functions on \mathfrak{Q}_0 , which is equipped with \mathfrak{Q} and \mathfrak{H} actions. The K_0 group of the algebra $\mathcal{C}^\infty \rtimes \mathfrak{H}$ is equal to the K_0 group of the orbifold \mathcal{Y} associated with the groupoid \mathfrak{H} . And the K_0 group of the twisted crossed product algebra $\tilde{\mathcal{C}}^\infty \rtimes_c (\hat{G} \rtimes \mathfrak{Q})$ is equal to the twisted K_c^0 -group [68] of the gerbe c on the orbifold $\hat{\mathcal{Y}}$ associated with the groupoid $\hat{G} \rtimes \mathfrak{Q}$.

Proposition 4.13. The K_0 -group of the G -gerbe $\mathcal{Y} = [\mathfrak{H}_0/\mathfrak{H}]$ is isomorphic to the K_c^0 -group of the $U(1)$ -gerbe c on the orbifold $\hat{\mathcal{Y}} = [\hat{G} \times \mathfrak{Q}_0/\mathfrak{Q}]$.

Brylinski and Nistor [19] proved that the cyclic homology of the groupoid algebra $\mathcal{C}^\infty \rtimes \mathfrak{H}$ is equal to the cohomology (with compact support) of the inertia orbifold $I\mathcal{Y}$ associated with the orbifold \mathcal{Y} . Tu and Xu [67] proved that the cyclic homology of the twisted groupoid algebra $\tilde{\mathcal{C}}^\infty \rtimes_c (\hat{G} \rtimes \Omega)$ is equal to the c -twisted cohomology (with compact support) of the inertia orbifold $I\hat{\mathcal{Y}}$ of $\hat{\mathcal{Y}}$. By the Chern character isomorphism, we have the following identification of cohomologies.

Proposition 4.14.

$$\bigoplus_{n \in \mathbb{Z}} H_{cpt}^{\bullet+2n}(I\mathcal{Y}, \mathbb{C}) \cong HP_\bullet(\mathcal{C}^\infty \rtimes \mathfrak{H}) \cong HP_\bullet(\tilde{\mathcal{C}}^\infty \rtimes_c (\hat{G} \rtimes \Omega)) \cong \bigoplus_{n \in \mathbb{Z}} H_{cpt}^{\bullet+2n}(I\hat{\mathcal{Y}}, c, \mathbb{C}).$$

We want to better understand the isomorphisms in Proposition 4.14 in the symplectic case. In what follows, we assume that there is a Ω -invariant symplectic form on Ω_0 . This happens when the base \mathcal{B} is a symplectic orbifold and the G -gerbe \mathcal{Y} and its dual $\hat{\mathcal{Y}}$ are equipped with symplectic structures pulled back from the one on \mathcal{B} . By choosing an Ω -invariant torsion free symplectic connection on Ω_0 , following [56], we consider the sheaf of deformation quantization $\mathcal{A}^{((\hbar))}$ on Ω_0 . $\mathcal{A}^{((\hbar))}$ is also a Ω -sheaf of algebras.

It follows from Theorem 4.8 that the algebras $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ and $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$ are also Morita equivalent. Consequently, the Hochschild cohomology of the algebra $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ is isomorphic to the Hochschild cohomology of the algebra $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$.

The Hochschild cohomology of $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ was computed in our joint work [56] with Pflaum and Posthuma, and it is equal to the de Rham cohomology $H^{\bullet-\ell}(I\mathcal{Y})((\hbar))$ of the inertia orbifold $I\mathcal{Y}$ with coefficients in $\mathbb{C}((\hbar))$ and a degree shifting ℓ given by the codimensions of the embeddings of components of $I\mathcal{Y}$ into \mathcal{Y} . We have the following computation for the Hochschild cohomology of the algebra $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$.

Theorem 4.15. *The Hochschild cohomology of the algebra $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$ is equal to the c -twisted de Rham cohomology of the orbifold $\hat{\mathcal{Y}}$ with coefficients in $\mathbb{C}((\hbar))$ and a shifting ℓ defined by the codimensions of embeddings of components of $I\hat{\mathcal{Y}}$ into $\hat{\mathcal{Y}}$. That is,*

$$HH^\bullet(\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega), \tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)) \cong H^{\bullet-\ell}(I\hat{\mathcal{Y}}, c)((\hbar)).$$

The proof of Theorem 4.15 will be given in Section 4.4. The Morita equivalence between $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ and $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$, which follows from Theorem 4.8, can now be combined with Theorem 4.15 to yield the following:

Theorem 4.16.

(1) *There are isomorphisms of cohomologies,*

$$\begin{aligned} H^{\bullet-\ell}(I\mathcal{Y})((\hbar)) &\cong HH^\bullet(\mathcal{C}^{((\hbar))} \rtimes \mathfrak{H}, \mathcal{C}^{((\hbar))} \rtimes \mathfrak{H}) \\ &\cong HH^\bullet(\tilde{\mathcal{C}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega), \tilde{\mathcal{C}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)) \cong H^{\bullet-\ell}(I\hat{\mathcal{Y}}, c)((\hbar)). \end{aligned}$$

(2) *Moreover, the above isomorphisms yield an isomorphism of graded vector spaces,*

$$H^{\bullet-\text{age}}(I\mathcal{Y})((\hbar)) \cong H^{\bullet-\text{age}}(I\hat{\mathcal{Y}}, c)((\hbar)),$$

where the vector spaces are equipped with the age grading as defined in Section 2.1.

Proof. The proof of (1) has been explained. Part (2) follows from Proposition 4.17 (4). \square

4.4. Quasi-isomorphism between Hochschild complexes. In this subsection, we explain the proofs of Theorem 4.15 and 4.16. We will show that our method of computation actually gives a slightly stronger result than what Theorem 4.15 states. More precisely, the Hochschild cochain complexes of $\mathcal{C}^{((h))} \rtimes \mathfrak{H}$ and $\mathcal{C}^{((h))} \times_c (\widehat{G} \rtimes \mathfrak{Q})$ form presheaves over the orbifold $\mathcal{B} = [\mathfrak{Q}_0/\mathfrak{Q}]$. We will show the following quasi-isomorphisms between (pre)sheaves and their properties.

Proposition 4.17.

- (1) *The (pre)sheaf of Hochschild cochain complexes of $\mathcal{C}^{((h))} \rtimes \mathfrak{H}$ is quasi-isomorphic to the sheaf of de Rham differential forms on $I\mathcal{Y}$ as (pre)sheaves over \mathcal{B} .*
- (2) *The (pre)sheaf of Hochschild cochain complexes of $\widetilde{\mathcal{C}}^{((h))} \times_c (\widehat{G} \rtimes \mathfrak{Q})$ is quasi-isomorphic to the sheaf of c -twisted de Rham differential forms on $I\widehat{\mathcal{Y}}$ as (pre)sheaves over \mathcal{B} .*
- (3) *The Morita equivalences constructed in Theorem 4.8 defines a quasi-isomorphism from the (pre)sheaf of Hochschild cochain complexes of $\mathcal{C}^{((h))} \rtimes \mathfrak{H}$ to the (pre)sheaf of Hochschild cochain complexes of $\widetilde{\mathcal{C}}^{((h))} \times_c (\widehat{G} \rtimes \mathfrak{Q})$ as (pre)sheaves over \mathcal{B} .*
- (4) *The sheaf of de Rham complex on $I\mathcal{Y}$ is quasi-isomorphic to the sheaf of c -twisted de Rham complex on $I\widehat{\mathcal{Y}}$ over \mathcal{B} . The quasi-isomorphism between sheaves of complexes is compatible with the filtration defined by the age function age on \mathcal{B} .*

Proof. We divide the proof into four parts according to the four corresponding statements (1)-(4).

Part I: This is proved in [56]. We explain the proof in more detail as a preparation for the generalization in the next step. See [56] for a complete treatment.

Let $\text{Bar}_\bullet(\mathcal{A}^{((h))} \rtimes \mathfrak{H})$ be the bar complex of the algebra $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$, and let $(\mathcal{A}^{((h))} \rtimes \mathfrak{H})^e$ be the enveloping algebra of $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$ as defined in Section 2.4 (see [56, Appendix] for more details). The Hochschild cochain complex of the algebra $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$ is

$$\text{Hom}_{(\mathcal{A}^{((h))} \rtimes \mathfrak{H})^e}(\text{Bar}_\bullet(\mathcal{A}^{((h))} \rtimes \mathfrak{H}), \mathcal{A}^{((h))} \rtimes \mathfrak{H}).$$

Let $\pi : \mathfrak{H}_0 \rightarrow \mathcal{Y} = [\mathfrak{H}_0/\mathfrak{H}]$ be the canonical projection map. We define a sheaf $\mathcal{S}^{((h))}$ on \mathcal{Y} by

$$\mathcal{S}^{((h))}(U) := C^\infty((\pi \circ s)^{-1}(U))((\hbar)),$$

where s is the source map on \mathfrak{H} . It is easy to check that $\mathcal{S}^{((h))}$ forms a sheaf, but the deformed convolution product is not well-defined on $\mathcal{S}^{((h))}$ because functions in $\mathcal{S}^{((h))}(U)$ may not have compact supports. This suggests that we should consider the sheaf $\mathcal{S}_{\text{cf}}^{((h))}$ defined by

$$\mathcal{S}_{\text{cf}}^{((h))}(U) := \{f \in C^\infty((\pi \circ s)^{-1}(U))((\hbar)) \mid \text{supp}(f) \cap (\pi \circ s)^{-1}(K) \text{ is compact for all compact } K \subset U\}.$$

It is easy to check that the deformed convolution product is well-defined on $\mathcal{S}_{\text{cf}}^{((h))}$ and makes $\mathcal{S}_{\text{cf}}^{((h))}$ into a sheaf of algebras over \mathcal{Y} .

Define a (pre)sheaf $\mathcal{H}_{\mathfrak{H}}^\bullet$ on \mathcal{Y} by

$$(4.7) \quad \mathcal{H}_{\mathfrak{H},h}^\bullet(U) := \text{Hom}((\mathcal{A}^{((h))}|_U \rtimes \mathfrak{H}|_U)^{\otimes \bullet}, \mathcal{S}_{\text{cf}}^{((h))}),$$

for any open subset U of \mathcal{Y} . In the above equation, Hom means bounded $\mathbb{C}((\hbar))$ -linear maps with respect to the bornologies on the algebras. In [56, Theorem I], it is proved that the natural inclusion map

$$\iota : C^\bullet(\mathcal{A}^{((h))} \rtimes \mathfrak{H}, \mathcal{A}^{((h))} \rtimes \mathfrak{H}) \rightarrow \mathcal{H}_{\mathfrak{H},h}^\bullet(\mathcal{Y})$$

is a quasi-isomorphism of cochain complexes compatible with the cup products.

It is not difficult to check that the (pre)sheaf $\mathcal{H}_{\mathfrak{H},h}^\bullet$ is fine, and therefore, we can use the Čech double complex of the (pre)sheaf $\mathcal{H}_{\mathfrak{H},h}^\bullet$ to compute the sheaf cohomology of the (pre)sheaf $\mathcal{H}_{\mathfrak{H},h}^\bullet$, which is isomorphic to the Hochschild cohomology of $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$ as an algebra. In particular, when we

choose a sufficiently fine covering of \mathcal{Y} , the Čech double complex degenerates at E_1 , so we can use the Čech cohomology of the sheaf $\mathcal{H}_{\mathfrak{H},h}$ to compute the Hochschild cohomology of $\mathcal{A}^{((h))} \rtimes \mathfrak{H}$. In [56, Section 4], we prove that on each open chart U the complex $\mathcal{H}_{\mathfrak{H},h}(U)$ is quasi-isomorphic to the de Rham complex of $I\mathcal{Y}|_U$ over U , where $I\mathcal{Y}|_U$ is the inertia orbifold associated to U . This proves that the Čech cohomology of the sheaf $\mathcal{H}_{\mathfrak{H},h}^\bullet$ is equal to the de Rham cohomology of the inertia orbifold $I\mathcal{Y}$.

Part II: We use the method developed in [56], which is recalled in Part I, to compute the Hochschild cohomology of $\tilde{\mathcal{A}}^{((h))} \rtimes_c (\hat{G} \rtimes \mathfrak{Q})$.

We consider the orbifold $\hat{\mathcal{Y}}$ presented by the groupoid $\hat{G} \rtimes \mathfrak{Q}$. Let $\pi : \hat{G} \times \mathfrak{Q}_0 \rightarrow \hat{\mathcal{Y}}$ be the canonical projection. Consider the sheaf

$$\mathcal{S}_{\text{cf},c}^{((h))}(U) := \{f \in C^\infty((\pi \circ s)^{-1}(U))((\hbar)) \mid \text{supp}(f) \cap (\pi \circ s)^{-1}(K) \text{ is compact for all compact } K \subset U\}.$$

It is easy to check that $\mathcal{S}_{\text{cf},c}^{((h))}$ with the c -twisted deformed convolution product, equation (4.5), defines a sheaf of algebras over $\hat{\mathcal{Y}}$. Define the (pre)sheaf of Hochschild complex over $\hat{\mathcal{Y}}$ by

$$(4.8) \quad \mathcal{H}_{\hat{G} \rtimes \mathfrak{Q},h}^\bullet(U) := \text{Hom}((\tilde{\mathcal{A}}^{((h))}|_U \rtimes (\hat{G} \rtimes \mathfrak{Q}|_U)^{\otimes \bullet}, \mathcal{S}_{\text{cf},c}^{((h))}(U))$$

for any open subset U of $\hat{\mathcal{Y}}$. The same arguments as [56, Section 3] prove that the natural inclusion map

$$\iota : C^\bullet(\tilde{\mathcal{A}}^{((h))} \rtimes_c (\hat{G} \rtimes \mathfrak{Q}), \tilde{\mathcal{A}}^{((h))} \rtimes_c (\hat{G} \rtimes \mathfrak{Q})) \rightarrow \mathcal{H}_{\hat{G} \rtimes \mathfrak{Q},h}^\bullet(\hat{\mathcal{Y}})$$

is a quasi-isomorphism of differential graded algebras.

Consider the sheaf $\mathcal{A}_{\hat{\mathcal{Y}}}^{((h))}$ on $\hat{\mathcal{Y}}$ whose sections over an open set $U \subset \hat{\mathcal{Y}}$ is defined by

$$\mathcal{A}_{\hat{\mathcal{Y}}}^{((h))}(U) := C^\infty((\pi)^{-1}(U))((\hbar))^\mathfrak{Q}.$$

It is easy to check that $\mathcal{A}_{\hat{\mathcal{Y}}}^{((h))}$ is a fine sheaf of commutative algebras over $\hat{\mathcal{Y}}$ with pointwise multiplication. Therefore, as a presheaf of modules of $\mathcal{A}_{\hat{\mathcal{Y}}}^{((h))}$, the presheaf $\mathcal{H}_{\hat{G} \rtimes \mathfrak{Q},h}^\bullet$ is fine. Therefore, we can use the Čech double complex to compute the cohomology of $\mathcal{H}_{\hat{G} \rtimes \mathfrak{Q},h}^\bullet(\hat{\mathcal{Y}})$. This allows us to reduce to local computations of $\mathcal{H}_{\hat{G} \rtimes \mathfrak{Q},h}^\bullet(U)$ for a sufficiently small open subset U of $\hat{\mathcal{Y}}$.

For $([\rho_0], x) \in \hat{G} \rtimes \mathfrak{Q}_0$, let $\mathfrak{Q}_{[\rho_0],x}$ be the isotropy group of the groupoid $\hat{G} \rtimes \mathfrak{Q}$ at $([\rho_0], x)$, which consists of arrows of the form $([\rho_0], q)$ with $q \in \mathfrak{Q}_x := \{q \in \mathfrak{Q}, t(q) = s(q) = x\}$ and $q([\rho_0]) = [\rho_0]$. Choose an open, connected neighborhood $W_{[\rho_0],q}$ for each $([\rho_0], q)$ in $\hat{G} \rtimes \mathfrak{Q}_{[\rho_0],x}$ such that s and t map $W_{[\rho_0],q}$ onto their images in $\hat{G} \times \mathfrak{Q}_0$ by diffeomorphisms. Define $M_{[\rho_0],x}$ to be the connected, open component of $\bigcap_{q \in \mathfrak{Q}_x, q([\rho_0]) = [\rho_0]} s(W_{[\rho_0],q}) \cap t(W_{[\rho_0],q})$. Define a $\mathfrak{Q}_{[\rho_0],x}$ action on $M_{[\rho_0],x}$ by

$$\mathfrak{Q}_{[\rho_0],x} \times M_{[\rho_0],x} \rightarrow M_{[\rho_0],x}, \quad (q, [\rho_0], x) \mapsto t(s^{-1}|_{[\rho_0],q}([\rho_0], x)).$$

In [56, Theorem IIIb.], we proved that the canonical inclusion induces a weak equivalence from the transformation groupoid $M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}$ to the groupoid $\hat{G} \rtimes \mathfrak{Q}|_{U_{[\rho_0],x}}$, where $U_{[\rho_0],x}$ is the image of $M_{[\rho_0],x}$ in $\hat{\mathcal{Y}}$ under the canonical projection π .

Now we restrict the $U(1)$ -valued cocycle c to the groupoid $M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}$ to get a cocycle denoted by $c_{[\rho_0],x}$. Following the proof of [56, Theorem III.b.], we can check that the twisted groupoid algebra $\tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}}(M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x})$ is Morita equivalent to $\tilde{\mathcal{A}}^{((h))}|_{U_{[\rho_0],x}} \rtimes_c (\hat{G} \rtimes \mathfrak{Q}|_{U_{[\rho_0],x}})$. Therefore, similar to [56, Theorem III.b.], the canonical cochain map

$$\mathcal{H}_{\hat{G} \rtimes \mathfrak{Q},h}^\bullet(U_{[\rho_0],x}) \rightarrow C^\bullet(\tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}}(M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}), \tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}}(M_{[\rho_0],x} \rtimes \mathfrak{Q}_{[\rho_0],x}))$$

is a quasi-isomorphism. This enables us to localize the computation of the Hochschild cohomology of the algebra $\tilde{\mathcal{A}}^{((h))} \rtimes_c (\widehat{G} \rtimes \Omega)$ to $\tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x}} \rtimes_{c_{[\rho_0],x}} (M_{[\rho_0],x} \rtimes \Omega_{[\rho_0],x})$.

We compute the Hochschild cohomology of $\tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x}} \rtimes_{c_{[\rho_0],x}} (M_{[\rho_0],x} \rtimes \Omega_{[\rho_0],x})$. Observe that since $c_{[\rho_0],x}$ is locally constant, the cocycle $c_{[\rho_0],x}$ is a lifting of a $U(1)$ -valued cocycle on $\Omega_{[\rho_0],x}$ via the natural projection $M_{[\rho_0],x} \rtimes \Omega_{[\rho_0],x} \rightarrow \Omega_{[\rho_0],x}$. To simplify notations, we assume that c is a $U(1)$ -valued 2-cocycle on a finite group Q that acts on a symplectic manifold P by symplectic diffeomorphisms. We want to compute the Hochschild cohomology of the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q$.

We follow the idea of Tu and Xu [67] to compute the Hochschild cohomology of $\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q$. As c is a $U(1)$ -valued 2-cocycle on a finite group Q , $[c]$ is a torsion element in $H^2(Q, U(1))$. This implies that there exists a sufficiently large integer m and a 1-cochain $\phi \in C^1(Q, U(1))$ such that $\tilde{c} = c\delta(\phi)$ is a 2-cocycle in $Z^2(Q, U(1))$ with δ as the group cohomology coboundary map and $\tilde{c}^m = 1$.

It is straightforward to check that the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q$ is isomorphic to $\tilde{\mathcal{A}}_P^{((h))} \rtimes_{\tilde{c}} Q$ via the map $\Upsilon_{P,Q} : \tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q \rightarrow \tilde{\mathcal{A}}_P^{((h))} \rtimes_{\tilde{c}} Q$ defined by $\Upsilon(f)(x, q) = \phi(q)^{-1} f(x, q)$. Hence, the Hochschild cohomology of the two algebras are naturally isomorphic. Without loss of generality, in the following, we assume that c takes value in $\mathbb{Z}/m\mathbb{Z}$.

As c is a $\mathbb{Z}/m\mathbb{Z}$ -valued 2-cocycle on Q , the cocycle c defines a central extension of Q :

$$1 \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rtimes_c Q \rightarrow Q \rightarrow 1.$$

This induces an extension of the following groupoids

$$1 \rightarrow P \times \mathbb{Z}/m\mathbb{Z} \rightarrow P \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q) \rightarrow P \rtimes Q \rightarrow 1,$$

where $\mathbb{Z}/m\mathbb{Z} \rtimes_c Q$ acts on P via the canonical group homomorphism $\mathbb{Z}/m\mathbb{Z} \rtimes_c Q \rightarrow Q$.

Consider the crossed product algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$. Notice that $\mathbb{Z}/m\mathbb{Z}$ acts on the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ by algebra automorphism, and the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q$ appears in $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ as the subspace of weight $\exp(\frac{2\pi\sqrt{-1}}{m})$ with respect to the $\mathbb{Z}/m\mathbb{Z}$ action. More precisely, we can easily check that the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ decomposes into a direct sum of subalgebras

$$(4.9) \quad \tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q) = \bigoplus_{k=0}^{m-1} \tilde{\mathcal{A}}_P^{((h))} \rtimes_{c^k} Q,$$

where c^k is an $\mathbb{Z}/m\mathbb{Z}$ -valued 2-cocycle on Q defined by k -th power of c . The isomorphism f_k from $\tilde{\mathcal{A}}_P^{((h))} \rtimes_{c^k} Q$ to $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ is defined by

$$f_k(\xi)(x, m_0, q) := \frac{1}{m} \exp\left(\frac{2m_0 k \pi \sqrt{-1}}{m}\right) \xi(x, q),$$

where $m_0 \in \mathbb{Z}/m\mathbb{Z}$.

The decomposition (4.9) of the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ naturally induces a decomposition of the Hochschild cohomology

$$\begin{aligned} & HH^\bullet(\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q), \tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)) \\ & \cong \bigoplus_{k=0}^{m-1} HH^\bullet(\tilde{\mathcal{A}}_P^{((h))} \rtimes_{c^k} Q, \tilde{\mathcal{A}}_P^{((h))} \rtimes_{c^k} Q). \end{aligned}$$

The above decomposition is taken with respect to the $\mathbb{Z}/m\mathbb{Z}$ action on the coefficient component of the Hochschild cohomology $HH^\bullet(\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q), \tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q))$. The Hochschild cohomology of the algebra $\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q$ is identified as the component with weight $\exp(\frac{2\pi\sqrt{-1}}{m})$.

With the above preparation, we can apply the result of [56, Theorem 4] to compute the Hochschild cohomology of $\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ to be

$$(4.10) \quad HH^\bullet(\tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q), \tilde{\mathcal{A}}_P^{((h))} \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)) \cong \bigoplus_{(\gamma) \in \mathbb{Z}/m\mathbb{Z} \rtimes_c Q} H_{Z(\gamma)}^{\bullet-\ell}(P^\gamma)((\hbar)),$$

where P^γ is the fixed submanifold of γ , and ℓ is a locally constant function measuring the codimension of P^γ in P , and $Z(\gamma)$ is the centralizer group of γ in $\mathbb{Z}/m\mathbb{Z} \rtimes_c Q$. By chasing through the quasi-isomorphisms constructed in [56, Section 4], we conclude that the above equation is actually compatible with the $\mathbb{Z}/m\mathbb{Z}$ action on both sides. We explain that the right hand side of equation (4.10) is defined by the cohomology of

$$\Omega^\bullet \left((P \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q))^{(0)} \right)^{\mathbb{Z}/m\mathbb{Z} \rtimes_c Q},$$

which has a natural $\mathbb{Z}/m\mathbb{Z}$ action as defined in [67, Lemma 4.13]. The component with weight $\exp(\frac{2\pi\sqrt{-1}}{m})$ of the left side of equation (4.10) is $HH^\bullet(\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q, \tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q)$. And by [67, equation (25)], the component with weight $\exp(\frac{2\pi\sqrt{-1}}{m})$ of the cohomology of $\Omega^\bullet \left((P \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q))^{(0)} \right)^{\mathbb{Z}/m\mathbb{Z} \rtimes_c Q}$ is the cohomology of $(\Pi_\gamma P^\gamma)/Q$ with values in the inner local system defined by c .

Hence, taking the components of weight $\exp(\frac{2\pi\sqrt{-1}}{m})$ on both sides of equation (4.10), we conclude that

$$HH^\bullet(\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q, \tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q) \cong H^{\bullet-\ell}(I\hat{\mathcal{Y}}, c)((\hbar)),$$

where $I\hat{\mathcal{Y}}$ is the inertia orbifold associated to the orbifold $\hat{\mathcal{Y}} = [P/Q]$.

Let L be a line bundle on $P \rtimes Q$ defined by $L := \mathbb{C} \times_{\mathbb{Z}/m\mathbb{Z}} (P \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q))$. According to [67, equation (23)], the algebra $\Omega^\bullet \left((P \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q))^{(0)} \right)^{\mathbb{Z}/m\mathbb{Z} \rtimes_c Q}$ of $(\mathbb{Z}/m\mathbb{Z} \rtimes_c Q)$ -invariant differential forms on $(P \rtimes (\mathbb{Z}/m\mathbb{Z} \rtimes_c Q))^{(0)}$ is decomposed into a direct sum of Q -invariant differential forms $(\Omega^\bullet((P \rtimes Q)^{(0)}, L'^{\otimes k})^Q, \nabla)$, where L' is the restriction of L on $B^{(0)}(P \rtimes Q)$, and $k = 0, \dots, m-1$, and ∇ is a unique connection determined by the cocycle c as explained in [67, Proposition 3.9].

In conclusion, we have shown that, locally, the Hochschild cochain complex of $\tilde{\mathcal{A}}_P^{((h))} \rtimes_c Q$ is quasi-isomorphic to the cochain complex $(\Omega^{\bullet-\ell}((P \rtimes Q)^{(0)}, L')^Q, \nabla)$ via a sequence I_c of natural quasi-isomorphisms constructed in [56, Section 4]. We can easily generalize this sequence together with its intermediate objects to write down a natural sequence, $I_{c[\rho_0],x}$, of cochain maps between the Hochschild cochain complex

$$C^\bullet(\tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x} \rtimes_{c[\rho_0],x} (M_{[\rho_0],x} \rtimes \Omega_{[\rho_0],x})}, \tilde{\mathcal{A}}^{((h))}|_{M_{[\rho_0],x} \rtimes_{c[\rho_0],x} (M_{[\rho_0],x} \rtimes \Omega_{[\rho_0],x})})$$

and

$$(\Omega^{\bullet-\ell}((M_{[\rho_0],x} \rtimes \Omega_{[\rho_0],x})^{(0)}, L'_{c[\rho_0],x})^{\Omega_{[\rho_0],x}}, \nabla^{c[\rho_0],x}).$$

Furthermore, we can use the cochain $\phi \in C^1(Q, U(1))$ to generalize the isomorphism $\Upsilon_{P,Q}$ to construct an isomorphism between the two sequences I_c and $I_{c[\rho_0],x}$. Therefore, we conclude that $I_{c[\rho_0],x}$ is a sequence of natural quasi-isomorphisms and glues together to define a sequence of quasi-isomorphisms between the presheaf $\mathcal{H}_{\hat{G} \rtimes_c \Omega, \hbar}^\bullet$ and the sheaf of the twisted de Rham complex $(\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar)), \nabla)$ as (pre)sheaves of algebras over $\hat{\mathcal{Y}}$. See [56, Thm. 4.17].

Therefore, we can conclude that the presheaf of the complex $\mathcal{H}_{\hat{G} \rtimes_c \Omega, \hbar}^\bullet$ over $\hat{\mathcal{Y}}$ is quasi-isomorphic to the sheaf of the twisted de Rham complex $(\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar)), \nabla)$ as (pre)sheaves of algebras over $\hat{\mathcal{Y}}$.

The algebraic structure on $(\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar)), \nabla)$ is locally defined to be

$$(4.11) \quad \alpha_1 \star \alpha_2|_{\gamma} = \sum_{\gamma_1, \gamma_2, \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2)} c(\gamma_1, \gamma_2) \iota_{\gamma_1}^*(\alpha_1) \wedge \iota_{\gamma_2}^*(\alpha_2),$$

where $\iota_{\gamma_i} : P^{\gamma_1, \gamma_2} = P^{\gamma_1} \cap P^{\gamma_2} \hookrightarrow P^{\gamma_i}, i = 1, 2$ is the canonical inclusion map of fixed point submanifolds. Hence, we have an isomorphism of algebras:

$$\left(HH^{\bullet}(\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega), \tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)), \cup \right) \cong (H^{\bullet-\ell}(I\hat{\mathcal{Y}}, c)((\hbar)), \star).$$

Remark 4.18. In this step, we have chosen to work locally with a $\mathbb{Z}/m\mathbb{Z}$ -valued 2-cocycle on Q to obtain the Hochschild cohomology of $\tilde{\mathcal{A}}_P^{((\hbar))} \rtimes Q$ via the trick of passing to the central extension $\mathbb{Z}/m\mathbb{Z} \rtimes_c Q$ as in [67]. One can take a more direct path by repeating the computations in [56, Section 4]. Because of the property that c is locally constant, methods in [56, Section 4] naturally generalize to compute the Hochschild cohomology of the algebra $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$.

Part III: Let \mathcal{B} be the orbifold defined by the quotient $[\Omega_0/\Omega]$. As a topological space, it is easy to see that \mathcal{B} is the same as \mathcal{Y} . Therefore, the (pre)sheaf $\mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet}$ of differential graded algebras is also a presheaf over \mathcal{B} . Similarly, the orbifold $\hat{\mathcal{Y}}$ is a fibration over \mathcal{B} with finite fibers. Therefore, the push-forward of the (pre)sheaf $\mathcal{H}_{\hat{G} \rtimes \Omega, \hbar}^{\bullet}$ defines a (pre)sheaf over \mathcal{B} . In this part, we want to show that the Morita equivalence bimodules \mathcal{M} and \mathcal{N} between $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}$ and $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)$ define quasi-isomorphisms between $\mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet}$ and $\mathcal{H}_{\hat{G} \rtimes \Omega, \hbar}^{\bullet}$ as (pre)sheaves over \mathcal{B} .

It is straightforward to see that the Morita equivalence bimodules \mathcal{M} and \mathcal{N} constructed in the proof of Theorem 4.8 are compatible with respect to localization to an open set U of \mathcal{B} . More precisely, let $\lambda_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{B}$ and $\lambda_{\hat{\mathcal{Y}}} : \hat{\mathcal{Y}} \rightarrow \mathcal{B}$ be the canonical projections, and let $\pi_{\mathcal{Y}}$ (respectively, $\pi_{\hat{\mathcal{Y}}}$) be the projection from $\mathfrak{H}_0 \rightarrow \mathcal{Y}$ (respectively $\hat{G} \times \Omega_0 \rightarrow \hat{\mathcal{Y}}$). It is not difficult to see that the restrictions of \mathcal{M} and \mathcal{N} to $(\lambda_{\mathcal{Y}} \circ \pi_{\mathcal{Y}} \circ s)^{-1}(U)$ define Morita equivalence bimodules between $\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}|_{\lambda_{\mathcal{Y}}^{-1}(U)}$ and $\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\hat{G} \rtimes \Omega)|_{\lambda_{\hat{\mathcal{Y}}}^{-1}(U)}$. Consequently, the Morita equivalence bimodules, together with the maps Ξ and Θ introduced in the proof of Theorem 4.8, induce quasi-isomorphisms between the cochain complexes $\mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet}$ and $\mathcal{H}_{\hat{G} \rtimes \Omega, \hbar}^{\bullet}$ as (pre)sheaves of differential graded algebras over \mathcal{B} .

We conclude from Part I and II that the sheaf of de Rham differential forms on $I\mathcal{Y}$ is quasi-isomorphic to the sheaf of c -twisted de Rham differential forms on $I\hat{\mathcal{Y}}$ viewed as sheaves of differential graded algebras over \mathcal{B} :

$$(\Omega_{I\mathcal{Y}}^{\bullet-\ell}((\hbar)), *) \simeq (\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar)), \star),$$

where $*$ on $\Omega_{I\mathcal{Y}}^{\bullet-\ell}$ is defined in the same way as \star in equation (4.11).

Part IV: We are left to show that the quasi-isomorphism

$$(4.12) \quad I : (\Omega_{I\mathcal{Y}}^{\bullet-\ell}((\hbar)), *) \simeq (\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar)), \star)$$

obtained in Part III is compatible with the filtration defined by the age function. Note that the age filtration (respectively, the codimension filtration ℓ) on $\Omega_{I\mathcal{Y}}^{\bullet}$ is determined by the age filtration (respectively, the codimension filtration) on $I\mathcal{B}$ via the map $\tilde{\lambda}_{\mathcal{Y}} : I\mathcal{Y} \rightarrow I\mathcal{B}$ since G acts on Ω_0 trivially. Similarly, the age filtration (respectively, the codimension filtration ℓ) on $I\hat{\mathcal{Y}}$ is determined by the age filtration (respectively, the codimension filtration) on $I\mathcal{B}$ via the fibration $\tilde{\lambda}_{\hat{\mathcal{Y}}} : I\hat{\mathcal{Y}} \rightarrow I\mathcal{B}$. We prove that the isomorphism I is compatible with the fibrations on $I\mathcal{Y}$ and $I\hat{\mathcal{Y}}$ over \mathcal{B} and conclude that I is compatible with the age filtration.

We can decompose the quasi-isomorphism I into the following sequences of isomorphisms:

$$\begin{aligned} H^{\bullet-\ell}(I\mathcal{Y})((\hbar)) &\stackrel{I_1}{\cong} HH^{\bullet}(\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}, \mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}) \\ &\stackrel{I_2}{\cong} HH^{\bullet}(\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega), \tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega)) \\ &\stackrel{I_3}{\cong} H^{\bullet-\ell}(I\widehat{\mathcal{Y}}, c)((\hbar)). \end{aligned}$$

The isomorphism I_1 between $HH^{\bullet}(\mathcal{A}^{((\hbar))} \rtimes \mathfrak{H}, \mathcal{A}^{((\hbar))} \rtimes \mathfrak{H})$ and $H^{\bullet-\ell}(I\mathcal{Y})((\hbar))$ is constructed in [56] and reviewed in Part I. The map I_2 from $HH^{\bullet}(\mathcal{C} \rtimes \mathfrak{H})$ to $HH^{\bullet}(\mathcal{C} \rtimes (\widehat{G} \rtimes_c \Omega))$ is a standard construction for Morita invariance of Hochschild cohomology from Theorem 4.8 as explained in Part III. Construction of an explicit quasi-isomorphism is explained in [56, Theorem A.12] and the isomorphism I_3 between $H^{\bullet-\ell}(I\widehat{\mathcal{Y}}, c)((\hbar))$ and $HH^{\bullet}(\tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega), \tilde{\mathcal{A}}^{((\hbar))} \rtimes_c (\widehat{G} \rtimes \Omega))$ is explained in Part II. In order to have an explicit formula for $I = I_3 \circ I_2 \circ I_1$, we need to write down the formulas for I_i , for $i = 1, 2, 3$.

Following Part III, we consider the presheaves $\mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet}$ (defined by equation (3.8)) and $\mathcal{H}_{\widehat{G} \rtimes \Omega, \hbar}^{\bullet}$ (defined by equation (4.8)) over the orbifold $\mathcal{B} = [\Omega_0/\Omega]$. We prove that the isomorphism I is realized by a sequence of quasi-isomorphisms of presheaves over \mathcal{B} :

$$\begin{aligned} \mathcal{I}_1 &: \mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet} \rightarrow (\Omega_{I\mathcal{Y}}^{\bullet-\ell}((\hbar)), d) \\ \mathcal{I}_2 &: \mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet} \rightarrow \mathcal{H}_{\widehat{G} \rtimes \Omega, \hbar}^{\bullet} \\ \mathcal{I}_3 &: \mathcal{H}_{\widehat{G} \rtimes \Omega, \hbar}^{\bullet} \rightarrow \Omega_{I\widehat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar), \nabla). \end{aligned}$$

Since \mathcal{I}_j , $j = 1, 2, 3$, are quasi-isomorphisms of presheaves, it suffices to look at their restrictions on a sufficiently small open set U of \mathcal{B} . On a sufficiently small open set U , the G -gerbe \mathcal{Y} over U can be represented by the groupoid extension

$$V \times G \rightarrow V \rtimes H \rightarrow V \rtimes Q,$$

so that the open set U is identified with the quotient $[V/Q]$. This is a special case of the global quotient considered in Section 4.1.

The map \mathcal{I}_1 is explained in [56, Section 4]. If α is a cocycle in $\mathcal{H}_{\mathfrak{H}, \hbar}^{\bullet}(U)$, then $\mathcal{I}_1(\alpha)$ is the image of the restriction of α on $I\mathcal{Y}$ in the cohomology of the double complex $\mathcal{C}_{\mathcal{X}}^{\bullet, \bullet}$ studied in [56, Proposition 4.8]. As is explained in Part II, equation (4.10), the map \mathcal{I}_3 is a $\mathbb{Z}/m\mathbb{Z}$ -equivariant version of the map \mathcal{I}_1 if we represent c by a cocycle in $\mathbb{Z}/m\mathbb{Z}$.

The map \mathcal{I}_2 is a standard map from Morita equivalence as is explained in [56, Theorem A.12]. We follow the construction in Section 3.3. Let ξ_i^{ρ} be a basis of V_{ρ} and η_{ρ}^i be the dual basis of V_{ρ}^* . If φ is a Hochschild cocycle on $\mathcal{A}_U^{((\hbar))} \rtimes H$, then $\mathcal{I}_2(\varphi)$ is a Hochschild cocycle on $(\mathcal{A}_U^{((\hbar))} \otimes C(\widehat{G})) \rtimes_c Q$ defined by

$$\begin{aligned} \mathcal{I}_2(\varphi)(a_1, \dots, a_k) &:= \\ \sum_{i_0, \dots, i_k, \rho_0, \dots, \rho_k} &\left(\frac{1}{\dim(V_{\rho})} \right)^k \Theta \left(\eta_{\rho_0}^{i_0}, \varphi(\Xi(\xi_{i_0}^{\rho_0}, a_1 \eta_{\rho_1}^{i_1}), \dots, \Xi(\xi_{i_{k-1}}^{\rho_{k-1}}, a_k \eta_{\rho_k}^{i_k})) \xi_{i_k}^{\rho_k} \right), \end{aligned}$$

where a_1, \dots, a_k are elements of $(\mathcal{A}_U^{((\hbar))} \otimes C(\widehat{G})) \rtimes_c Q$. Here, $C(\widehat{G})$ is the algebra of functions on the finite set \widehat{G} . The algebra $(\mathcal{A}_U^{((\hbar))} \otimes C(\widehat{G})) \rtimes_c Q$ can be identified as the twisted crossed product algebra of deformation quantization on $U \times \widehat{G}$ by the Q -action, namely, the quantized twisted convolution algebra $\tilde{A}_{U \times \widehat{G}}^{((\hbar))} \rtimes_c Q$.

Notice that $\xi_{i_{p-1}}^{\rho_{p-1}}$ (respectively, $\eta_{\rho_p}^{i_p}$) are supported only on the identity component of $(\tilde{\mathcal{A}}_U \otimes \mathcal{V}_G) \rtimes_c Q$ (respectively, $(\tilde{\mathcal{A}}_U \otimes \mathcal{V}_G^*) \rtimes_c Q$). If we choose a_1, \dots, a_k to be functions supported on the identity component of $\tilde{\mathcal{A}}_U \rtimes_c Q$, then by equation (4.6), $\Xi(\xi_{i_{p-1}}^{\rho_{p-1}}, a_p \eta_{\rho_p}^{i_p})$ vanishes if $\rho_p \neq \rho_{p-1}$ and is equal to $a_p([\rho]) \Xi(\xi_{i_{p-1}}^{\rho_{p-1}}, \eta_{\rho_p}^{i_p})$ for any $p = 1, \dots, k$. Using the fact that elements in G act trivially on U and the relation $\eta_i^\rho(\xi_i^\rho) = 1$, we see that when a_1, \dots, a_k are supported on the identity component, $I_2(\varphi)(a_1, \dots, a_k)$ is equal to

$$(4.13) \quad \begin{aligned} & \sum_{i, \rho} \frac{1}{\dim(V_\rho)} \Theta(\eta_\rho^i, \varphi(a_1([\rho]), \dots, a_k([\rho])) \xi_i^\rho) \\ &= \sum_{i, \rho, g, q: q([\rho]) = [\rho]} \frac{1}{\dim(V_\rho)} \varphi(a_1([\rho]), \dots, a_k([\rho]))(g, q) \operatorname{tr}(\rho(g)(T_q^{[\rho]})^{-1}). \end{aligned}$$

We observe that the maps \mathcal{I}_1 and \mathcal{I}_3 only count the information of the cocycle φ along the space U direction, which is completely determined by the part $\varphi(a_1([\rho]), \dots, a_k([\rho]))(g, q)$ in equation (4.13). Therefore, we conclude that the map $I = \mathcal{I}_3 \circ \mathcal{I}_2 \circ (\mathcal{I}_1^{-1})$ on a cohomology class α of $\Omega_{I\mathcal{Y}}^{\bullet-\ell}(U)((\hbar))$ can be expressed by

$$(4.14) \quad I(\alpha) = \sum_{g, q, \rho: q([\rho]) = [\rho]} \frac{1}{\dim(V_\rho)} \alpha(g, q) \operatorname{tr}(\rho(g)(T_q^{[\rho]})^{-1})([\rho], q),$$

where we have written $\alpha = \sum_{g, q} \alpha(g, q) \in \Omega^*(U^{g, q})$, which is invariant under the conjugacy action of H .

Now extending the expression of I in equation (4.14) to the whole orbifold, we have the following isomorphism. We represent a cohomology class α on $I\mathcal{Y}$ as $\alpha = \sum_{g, q: s(q)=t(q)} \alpha(g, q)$ such that α is a closed differential form on $\mathfrak{H}^{(0)} = \{(g, q) \in \mathfrak{H} : s(q) = t(q)\} \subset \mathfrak{H}$ that is invariant under the conjugacy action of \mathfrak{H} on $\mathfrak{H}^{(0)}$. Then $I(\alpha)$ can be written as a differential form supported on $\mathfrak{Q}' := \{([\rho], q) : q([\rho]) = [\rho], s(q) = t(q)\} \subset \hat{G} \rtimes_c \mathfrak{Q}$ that is invariant under the c -twisted conjugation action by $\hat{G} \rtimes_c \mathfrak{Q}$,

$$(4.15) \quad I(\alpha)([\rho], q) = \sum_g \frac{1}{\dim(V_\rho)} \alpha(g, q) \operatorname{tr}(\rho(g) T_q^{[\rho]})^{-1}.$$

As stated in Remark 3.4, the above formula for the map I is a full generalization of the map I in Propositions 3.4-3.5 when $\mathcal{B} = BQ$.

Similar to Proposition 3.4, the above expression (4.14) shows that, locally, the quasi-isomorphism I is compatible with respect to the conjugacy classes of the group Q_U . More explicitly, let $U^{\langle q \rangle}$ be the component of the inertia orbifold $I\mathcal{U} \subset I\mathcal{B}$ defined by the conjugacy class $\langle q \rangle \subset Q_U$ and $\Omega_{I\mathcal{Y}}^{\bullet-\ell}(U)((\hbar))|_{U^{\langle q \rangle}}$ (and $\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(U)((\hbar))|_{U^{\langle q \rangle}}$) be the space of differential forms on $I\mathcal{Y}$ (and on $I\hat{\mathcal{Y}}$) supported on $\tilde{\lambda}_{\mathcal{Y}}^{-1}(U^{\langle q \rangle})$ (and $\tilde{\lambda}_{\hat{\mathcal{Y}}}^{-1}(U^{\langle q \rangle})$). The isomorphism I in (4.14) defines a quasi-isomorphism

$$I|_{U^{\langle q \rangle}} : (\Omega_{I\mathcal{Y}}^{\bullet-\ell}(U)((\hbar))|_{U^{\langle q \rangle}}, d) \longrightarrow (\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\ell}(L')((\hbar))|_{U^{\langle q \rangle}}, \nabla).$$

As both the age function and codimension function ℓ on $I\mathcal{Y}$ and $I\hat{\mathcal{Y}}$ are determined by their corresponding functions on $I\mathcal{B}$, we conclude that the quasi-isomorphism I in (4.14) also induces a quasi-isomorphism

$$(\Omega_{I\mathcal{Y}}^{\bullet-\text{age}}(U)((\hbar))|_{U^{\langle q \rangle}}, d) \longrightarrow (\Omega_{I\hat{\mathcal{Y}}}^{\bullet-\text{age}}(L')((\hbar))|_{U^{\langle q \rangle}}, \nabla).$$

This shows that the quasi-isomorphism (4.12) induces a quasi-isomorphism on the shifted complexes by the age function,

$$(\Omega_{I\mathcal{Y}}^{\bullet-\text{age}}((\hbar)), d) \simeq (\Omega_{I\widehat{\mathcal{Y}}}^{\bullet-\text{age}}(L')((\hbar)), \nabla).$$

Globalizing the above local arguments via Čech arguments, and noticing that all the construction above is canonical as morphisms of (pre)sheaves over \mathcal{B} , we conclude with the following equality of vector spaces:

$$H_{\text{CR}}^{\bullet}(\mathcal{Y}, \mathbb{C}((\hbar))) \stackrel{\text{def}}{=} H^{\bullet-\text{age}}(I\mathcal{Y}, \mathbb{C}((\hbar))) \cong H^{\bullet-\text{age}}(I\widehat{\mathcal{Y}}, c, \mathbb{C}((\hbar))) \stackrel{\text{def}}{=} H_{\text{orb}}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C}((\hbar))).$$

□

Remark 4.19. *It can be seen from the explicit description of the quasi-isomorphism I in (4.15) that I induces an isomorphism between cohomologies with \mathbb{C} coefficients, namely,*

$$H_{\text{CR}}^{\bullet}(\mathcal{Y}, \mathbb{C}) \simeq H_{\text{orb}}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C}).$$

5. CHEN-RUAN ORBIFOLD COHOMOLOGY RINGS

In this section, we discuss results on the Chen-Ruan orbifold cohomology rings of étale gerbes.

5.1. Review of Chen-Ruan cohomology. This section contains a summary of the Chen-Ruan orbifold cohomology ring [23] and twisted orbifold cohomology rings [59].

Throughout this section, let \mathcal{X} be a compact almost complex orbifold. The *inertia orbifold* of \mathcal{X} is the orbifold $I\mathcal{X}$ whose points are pairs $(x, (g))$ where $x \in \mathcal{X}$ and $(g) \subset \text{Iso}(x)$ is a conjugacy class of the isotropy subgroup of the point $x \in \mathcal{X}$. There is a natural map

$$\mathbf{p}_{\mathcal{X}} : I\mathcal{X} \rightarrow \mathcal{X}, \quad (x, (g)) \mapsto x.$$

The inertia orbifold $I\mathcal{X}$ is, in general, disconnected; let

$$I\mathcal{X} = \coprod_{i \in \mathcal{I}} \mathcal{X}_i$$

be the decomposition into connected components, where \mathcal{I} is an index set.

The following construction will be used subsequently.

Definition 5.1. Let K be a finite group acting on a vector space V . Let $k \in K$ be an element of order r . Consider the decomposition of V into k -eigenspaces,

$$V = \bigoplus_{i=0}^{r-1} V_i,$$

where k acts on V_i with eigenvalue $\exp(\frac{2\pi\sqrt{-1}i}{r})$. Define the *age* function by

$$\text{age}_V(k) := \sum_{i=1}^{r-1} \frac{i}{r} \dim V_i \in \mathbb{Q}.$$

For $(x, (g)) \in I\mathcal{X}$, it follows that the group element g acts on the tangent space $T_x\mathcal{X}$. Set

$$\text{age}((x, (g))) := \text{age}_{T_x\mathcal{X}}(g).$$

It is easy to see that $\text{age}((x, (g)))$ defines a locally constant function $\text{age} : I\mathcal{X} \rightarrow \mathbb{Q}$. Let $\text{age}(\mathcal{X}_i)$ be its value on the connected component \mathcal{X}_i .

Definition 5.2. The *Chen-Ruan orbifold cohomology groups* of \mathcal{X} are defined to be the cohomology groups of the inertia orbifold,

$$H_{CR}^\bullet(\mathcal{X}, \mathbb{C}) := H^\bullet(I\mathcal{X}, \mathbb{C}) = \oplus_{i \in \mathcal{I}} H^\bullet(\mathcal{X}_i, \mathbb{C}).$$

The grading used in the Chen-Ruan cohomology is the so-called *age-grading*: for a class $\alpha \in H^p(\mathcal{X}_i, \mathbb{C})$, its degree as a class in $H_{CR}^\bullet(\mathcal{X}, \mathbb{C})$ is $p + 2\text{age}(\mathcal{X}_i)$.

Remark 5.3. Here, we use the field \mathbb{C} of complex numbers as coefficients for the cohomology groups. Other fields can (and will) be used as coefficients.

The Chen-Ruan cohomology $H_{CR}^\bullet(\mathcal{X}, \mathbb{C})$ is equipped with a non-degenerate pairing called the *orbifold Poincaré pairing*, which is constructed as follows. There is an isomorphism $I_{\mathcal{X}} : I\mathcal{X} \rightarrow I\mathcal{X}$ given by $(x, (g)) \mapsto (x, (g^{-1}))$. Clearly, the composition $I_{\mathcal{X}} \circ I_{\mathcal{X}}$ is the identity map. A component \mathcal{X}_i is mapped isomorphically to a component we denote by \mathcal{X}_{iI} .

Definition 5.4. The orbifold Poincaré pairing $(-, -)_{orb}^{\mathcal{X}}$ is defined as follows. For $\alpha \in H^\bullet(\mathcal{X}_i, \mathbb{C})$ and $\beta \in H^\bullet(\mathcal{X}_{iI}, \mathbb{C})$, define

$$(\alpha, \beta)_{orb}^{\mathcal{X}} := \int_{\mathcal{X}_i} \alpha \cup I_{\mathcal{X}}^* \beta.$$

The pairing $(-, -)_{orb}^{\mathcal{X}}$ is extended to the whole $H^\bullet(I\mathcal{X}, \mathbb{C})$ by requiring bilinearity.

Remark 5.5. The orbifold Poincaré pairing differs from the Poincaré pairing on the cohomology $H^\bullet(I\mathcal{X}, \mathbb{C})$ because of the factor $I_{\mathcal{X}}^*$.

The Chen-Ruan cohomology $H_{CR}^\bullet(\mathcal{X}, \mathbb{C})$ is also equipped with a new product structure called the *Chen-Ruan orbifold cup product*. We briefly recall its construction. Let $\mathcal{X}_{(2)}$ be the 2-multi-sector of \mathcal{X} . It can be understood as the space whose points are $(x, (g, h))$, where $x \in \mathcal{X}$, $g, h \in \text{Iso}(x)$, and

$$(g, h) := \{(kgk^{-1}, khk^{-1}) | k \in \text{Iso}(x)\} \subset \text{Iso}(x) \times \text{Iso}(x)$$

is a *biconjugacy class* of $\text{Iso}(x)$. There are three *evaluation maps*:

$$\begin{aligned} ev_{\mathcal{X},1} : \mathcal{X}_{(2)} &\rightarrow I\mathcal{X}, & (x, (g, h)) &\mapsto (x, (g)); \\ ev_{\mathcal{X},2} : \mathcal{X}_{(2)} &\rightarrow I\mathcal{X}, & (x, (g, h)) &\mapsto (x, (h)); \\ ev_{\mathcal{X},3} : \mathcal{X}_{(2)} &\rightarrow I\mathcal{X}, & (x, (g, h)) &\mapsto (x, ((gh)^{-1})). \end{aligned}$$

There is also another natural map that forgets the biconjugacy class:

$$p_{\mathcal{X}_{(2)}} : \mathcal{X}_{(2)} \rightarrow \mathcal{X}, \quad (x, (g, h)) \mapsto x.$$

The key ingredient in the Chen-Ruan cup product is the so-called *obstruction bundle*

$$Ob_{\mathcal{X}} \rightarrow \mathcal{X}_{(2)}.$$

The original construction of $Ob_{\mathcal{X}}$ in [23] involves the moduli spaces of genus-0 degree-0 orbifold stable maps to \mathcal{X} , which is complicated. The construction has since been simplified. We present two descriptions.

Construction 5.1 (see [37], Theorem 2). Let $(x, (g, h)) \in \mathcal{X}_{(2)}$. The group $\langle g, h \rangle$ generated by g, h acts on the fiber $p_{\mathcal{X}_{(2)}}^* T\mathcal{X}|_{(x, (g, h))} = T_x \mathcal{X}$, yielding a decomposition $p_{\mathcal{X}_{(2)}}^* T\mathcal{X}|_{(x, (g, h))} = \bigoplus V_i \otimes T_i$, where the V_i 's are irreducible representations of $\langle g, h \rangle$. Varying the point $(x, (g, h)) \in \mathcal{X}_{(2)}$ in a connected component of $\mathcal{X}_{(2)}$ yields a global decomposition of vector bundles over this component:

$$p_{\mathcal{X}_{(2)}}^* T\mathcal{X} = \bigoplus V_i \otimes T_i,$$

where T_i 's are vector bundles over this component. Then the restriction of $Ob_{\mathcal{X}}$ to this component of $\mathcal{X}_{(2)}$ is equal to

$$\oplus T_i^{\oplus h_i},$$

where

$$(5.1) \quad h_i = \text{age}_{V_i}(g) + \text{age}_{V_i}(h) - \text{age}_{V_i}(gh) + \dim V_i^{\langle g, h \rangle} - \dim V_i^{gh}.$$

The terms $V_i^{\langle g, h \rangle}$ and V_i^{gh} are the vector subspaces fixed by $\langle g, h \rangle$ and gh respectively. As stated in [37], it can be shown that the numbers h_i are nonnegative integers, so the above equation makes sense.

Construction 5.2 (see [39]). Consider the component \mathcal{X}_i . At any point $(x, (g)) \in \mathcal{X}_i$ the group $\langle g \rangle$ acts on the fiber $\mathbf{p}_{\mathcal{X}}^* T\mathcal{X}|_{(x, (g))} = T_x \mathcal{X}$. The decomposition into g -eigenspaces can be globalized:

$$(\mathbf{p}_{\mathcal{X}}^* T\mathcal{X})|_{\mathcal{X}_i} = \bigoplus_k W_{i,k}^{\mathcal{X}}$$

where $W_{i,k}^{\mathcal{X}}$ is the eigen-bundle on which g acts with eigenvalue $\exp(\frac{2\pi\sqrt{-1}k}{r})$ and r is the order of g . Put

$$S_i^{\mathcal{X}} := \bigoplus_{k \neq 0} \frac{k}{r} W_{i,k}^{\mathcal{X}}.$$

This is an element in the K -theory of \mathcal{X}_i .

Consider the locus $\mathcal{X}_{i_1, i_2} := \text{ev}_{\mathcal{X}, 1}^{-1}(\mathcal{X}_{i_1}) \cap \text{ev}_{\mathcal{X}, 2}^{-1}(\mathcal{X}_{i_2})$. Then the image $\text{ev}_{\mathcal{X}, 3}(\mathcal{X}_{i_1, i_2})$ is contained in a component of $I\mathcal{X}$ denoted by \mathcal{X}_{i_3} . The K -theory class of the restriction of $Ob_{\mathcal{X}}$ to \mathcal{X}_{i_1, i_2} is given by

$$(5.2) \quad Ob_{\mathcal{X}}|_{\mathcal{X}_{i_1, i_2}} = T\mathcal{X}_{i_1, i_2} \ominus \mathbf{p}_{\mathcal{X}_{(2)}}^* T\mathcal{X}|_{\mathcal{X}_{i_1, i_2}} \oplus \bigoplus_{j=1}^3 \text{ev}_{\mathcal{X}, j}^* S_{i_j}^{\mathcal{X}}.$$

We now come to the definition of the Chen-Ruan cup product. Let $e(-)$ denote the Euler class.

Definition 5.6. For classes $\alpha_1, \alpha_2, \alpha_3 \in H^{\bullet}(I\mathcal{X}, \mathbb{C})$, define

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{\mathcal{X}} := \int_{\mathcal{X}_{(2)}} \text{ev}_{\mathcal{X}, 1}^* \alpha_1 \cup \text{ev}_{\mathcal{X}, 2}^* \alpha_2 \cup \text{ev}_{\mathcal{X}, 3}^* \alpha_3 \cup e(Ob_{\mathcal{X}}).$$

Fix an additive basis $\{\phi_i\}$ of $H^{\bullet}(I\mathcal{X}, \mathbb{C})$ such that each element ϕ_i is homogeneous and is supported on one connected component of $I\mathcal{X}$. Let $\phi^i := PD(\phi_i)$ be the class dual to ϕ_i under the orbifold Poincaré pairing $(-, -)_{orb}^{\mathcal{X}}$. The Chen-Ruan cup product of $\alpha_1, \alpha_2 \in H_{CR}^{\bullet}(\mathcal{X}, \mathbb{C})$ is defined as

$$(5.3) \quad \alpha_1 \star_{orb} \alpha_2 := \sum_i \langle \alpha_1, \alpha_2, \phi_i \rangle^{\mathcal{X}} \phi^i.$$

It follows from the definition that $(\alpha_1 \star_{orb} \alpha_2, \phi_i)_{orb}^{\mathcal{X}} = \langle \alpha_1, \alpha_2, \phi_i \rangle^{\mathcal{X}}$. The Chen-Ruan orbifold cohomology $H_{CR}^{\bullet}(\mathcal{X}, \mathbb{C})$ equipped with the above structures is a graded (super-)commutative \mathbb{C} -algebra.

Next, we recall the construction of the *twisted orbifold cohomology* [59]. Let \mathcal{X} be a compact almost complex orbifold as above. Let c be a flat $U(1)$ -gerbe on \mathcal{X} . It follows from the discussion in [55] that given such a $U(1)$ -gerbe c , one can construct a *inner local system* [59], which is a line bundle

$$\mathcal{L}_c \rightarrow I\mathcal{X}$$

satisfying various properties that are explained in [59, Definition 3.1]. The c -twisted orbifold cohomology groups are defined to be

$$H_{orb}^{\bullet}(\mathcal{X}, c, \mathbb{C}) := H^{\bullet}(I\mathcal{X}, \mathcal{L}_c, \mathbb{C}),$$

namely, the cohomology groups of $I\mathcal{X}$ with coefficients in the inner local system \mathcal{L}_c . The groups $H_{orb}^{\bullet}(\mathcal{X}, c, \mathbb{C})$ are equipped with the age-grading that is defined in the same way as Definition 5.2.

The definition of a c -twisted orbifold Poincaré pairing for $H_{orb}^\bullet(\mathcal{X}, c, \mathbb{C})$ is exactly parallel to Definition 5.4; for $\alpha \in H^\bullet(\mathcal{X}_i, \mathcal{L}_c)$ and $\beta \in H^\bullet(\mathcal{X}_{iI}, \mathcal{L}_c)$, define

$$(\alpha, \beta)_{orb, c}^\mathcal{X} := \int_{\mathcal{X}_i} \alpha \cup I_{\mathcal{X}}^* \beta.$$

The above integral makes sense since there exists a canonical trivialization¹³ of the line bundle $(\mathcal{L}_c|_{\mathcal{X}_i}) \otimes I_{\mathcal{X}}^*(\mathcal{L}_c|_{\mathcal{X}_{iI}})$. This is a property of the inner local system [59, Definition 3.1 (1)].

The c -twisted orbifold cohomology $H_{orb}^\bullet(\mathcal{X}, c, \mathbb{C})$ also carries an orbifold cup product \star_c defined using the obstruction bundle $Ob_{\mathcal{X}}$. The definition is parallel to Definition 5.6. For classes $\alpha_1, \alpha_2, \alpha_3 \in H^\bullet(I\mathcal{X}, \mathcal{L}_c, \mathbb{C})$, define

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_c^\mathcal{X} := \int_{\mathcal{X}_{(2)}} ev_{\mathcal{X}, 1}^* \alpha_1 \cup ev_{\mathcal{X}, 2}^* \alpha_2 \cup ev_{\mathcal{X}, 3}^* \alpha_3 \cup e(Ob_{\mathcal{X}}).$$

The integral above makes sense because there exists a canonical trivialization¹⁴ of $\otimes_{j=1}^3 ev_{\mathcal{X}, j}^*(\mathcal{L}_c|_{\mathcal{X}_{i_j}})$, which is again due to properties of the inner local system [59, Definition 3.1 (3)].

Fix an additive basis $\{\phi_i\}$ of $H^\bullet(I\mathcal{X}, \mathcal{L}_c, \mathbb{C})$ such that each element ϕ_i is homogeneous and is supported on one connected component of $I\mathcal{X}$. Let $\phi^i := PD(\phi_i)$ be the class dual to ϕ_i under the pairing $(-, -)_{orb, c}^\mathcal{X}$. The c -twisted orbifold cup product of $\alpha_1, \alpha_2 \in H_{orb}^\bullet(\mathcal{X}, c, \mathbb{C})$ is defined as

$$(5.4) \quad \alpha_1 \star_c \alpha_2 := \sum_i \langle \alpha_1, \alpha_2, \phi_i \rangle_c^\mathcal{X} \phi^i.$$

It follows from the definition that $(\alpha_1 \star_{orb} \alpha_2, \phi_i)_{orb, c}^\mathcal{X} = \langle \alpha_1, \alpha_2, \phi_i \rangle_c^\mathcal{X}$.

The c -twisted orbifold cohomology $H_{orb}^\bullet(\mathcal{X}, c, \mathbb{C})$ equipped with the above structures is a graded (super-)commutative \mathbb{C} -algebra.

5.2. Chen-Ruan cohomology of étale gerbes. Let \mathcal{B} be a compact connected almost complex orbifold, and G be a finite group. Let $\mathcal{Y} \rightarrow \mathcal{B}$ be a G -gerbe over \mathcal{B} and $\widehat{\mathcal{Y}} \rightarrow \mathcal{B}$ be its dual, equipped with the flat $U(1)$ -gerbe c . Denote by \mathcal{L}_c the inner local system associated with c .

The dual $\widehat{\mathcal{Y}}$ is disconnected; let

$$\widehat{\mathcal{Y}} = \coprod_{i \in \mathbf{I}} \widehat{\mathcal{Y}}_i$$

be the decomposition into connected components. Here, \mathbf{I} is an index set. Let c_i be the $U(1)$ -gerbe on $\widehat{\mathcal{Y}}_i$ obtained by restricting c to $\widehat{\mathcal{Y}}_i$, and let \mathcal{L}_{c_i} be the inner local system associated to c_i . By definition the c -twisted orbifold cohomology $H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ is a direct sum

$$(5.5) \quad H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C}) = \bigoplus_{i \in \mathbf{I}} H_{orb}^\bullet(\widehat{\mathcal{Y}}_i, c_i, \mathbb{C}).$$

The ring structures and pairings are compatible with this decomposition: if $\alpha_1, \alpha_2, \alpha_3 \in H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ are decomposed with respect to (5.5) as

$$(5.6) \quad \alpha_1 = \oplus_i \alpha_{1i}, \alpha_2 = \oplus_i \alpha_{2i}, \alpha_3 = \oplus_i \alpha_{3i}, \quad \alpha_{1i}, \alpha_{2i}, \alpha_{3i} \in H_{orb}^\bullet(\widehat{\mathcal{Y}}_i, c_i, \mathbb{C}),$$

¹³For simplicity, we omit this trivialization in our notation.

¹⁴We again omit the trivialization from our notation.

then

$$\begin{aligned}
 \langle \alpha_1, \alpha_2, \alpha_3 \rangle_c^{\widehat{\mathcal{Y}}} &= \sum_{\mathbf{i}} \langle \alpha_{1\mathbf{i}}, \alpha_{2\mathbf{i}}, \alpha_{3\mathbf{i}} \rangle_{c_{\mathbf{i}}}^{\widehat{\mathcal{Y}}_{\mathbf{i}}}, \\
 \alpha_1 \star_c \alpha_2 &= \oplus_{\mathbf{i}} (\alpha_{1\mathbf{i}} \star_{c_{\mathbf{i}}} \alpha_{2\mathbf{i}}), \\
 (\alpha_1, \alpha_2)_{orb, c}^{\widehat{\mathcal{Y}}} &= \sum_{\mathbf{i}} (\alpha_{1\mathbf{i}}, \alpha_{2\mathbf{i}})_{orb, c_{\mathbf{i}}}^{\widehat{\mathcal{Y}}_{\mathbf{i}}}.
 \end{aligned}
 \tag{5.7}$$

Suppose that \mathcal{B} is symplectic and is equipped with a compatible almost complex structure. We equip \mathcal{Y} and $\widehat{\mathcal{Y}}$ with the symplectic and compatible almost complex structures induced from those on \mathcal{B} . In Section 4.3, we have constructed an additive isomorphism

$$\Psi := I^{-1} : H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C}) \rightarrow H_{CR}^{\bullet}(\mathcal{Y}, \mathbb{C}),$$

which respects the age-gradings. See Proposition 4.14 and Theorem 4.16. In this section, we analyze the compatibility of Ψ with the cup products \star_{orb} and \star_c . The main result, Theorem 5.10, states that Ψ is in fact a ring isomorphism.

We begin by comparing the obstruction bundles. Consider the natural maps between 2-multi-sectors

$$\pi_{\mathcal{Y}} : \mathcal{Y}_{(2)} \rightarrow \mathcal{B}_{(2)}, \quad \pi_{\widehat{\mathcal{Y}}} : \widehat{\mathcal{Y}}_{(2)} \rightarrow \mathcal{B}_{(2)},$$

which are induced from the maps $\mathcal{Y} \rightarrow \mathcal{B}$ and $\widehat{\mathcal{Y}} \rightarrow \mathcal{B}$.

Proposition 5.7. *The following relations hold among obstruction bundles.*

$$Ob_{\mathcal{Y}} = \pi_{\mathcal{Y}}^* Ob_{\mathcal{B}}, \tag{5.8}$$

$$Ob_{\widehat{\mathcal{Y}}} = \pi_{\widehat{\mathcal{Y}}}^* Ob_{\mathcal{B}}. \tag{5.9}$$

The proof is an easy application of the constructions of obstruction bundles.

Proof of (5.8). Denote by $f : \mathcal{Y} \rightarrow \mathcal{B}$ the structure map of the G -gerbe. The map f is étale and $T\mathcal{Y} = f^*T\mathcal{B}$. Consider the situation of Construction 5.1. Let $(x, (h_1, h_2)) \in \mathcal{Y}_{(2)}$ and let $(x, (q_1, q_2)) \in \mathcal{B}_{(2)}$ be its image under $\pi_{\mathcal{Y}}$. Then we have a decomposition $\mathbf{p}_{\mathcal{B}_{(2)}}^* T\mathcal{B}|_{(x, (g, h))} = \bigoplus V_i \otimes T_i$ of vector bundles, where the V_i 's are irreducible representations of the group $\langle q_1, q_2 \rangle$ and the T_i 's are vector bundles over the component of $\mathcal{B}_{(2)}$ containing $(x, (q_1, q_2))$. Since $f : \mathcal{Y} \rightarrow \mathcal{B}$ is a G -gerbe, the induced group homomorphism $\langle h_1, h_2 \rangle \rightarrow \langle q_1, q_2 \rangle$, given by $h_1 \mapsto q_1, h_2 \mapsto q_2$, is surjective. Hence, the V_i 's can be viewed as irreducible representations of $\langle h_1, h_2 \rangle$. Also, the group G acts trivially on fibers of $T\mathcal{Y}$. Also note that $\mathbf{p}_{\mathcal{B}_{(2)}} \circ \pi_{\mathcal{Y}} = f \circ \mathbf{p}_{\mathcal{Y}_{(2)}}$. By this discussion, it follows that the decomposition of the bundle $\mathbf{p}_{\mathcal{Y}_{(2)}}^* T\mathcal{Y}$ as in Construction 5.1 is given by

$$\mathbf{p}_{\mathcal{Y}_{(2)}}^* T\mathcal{Y} = \bigoplus V_i \otimes \pi_{\mathcal{Y}}^* T_i.$$

According to Construction 5.1, we have

$$Ob_{\mathcal{Y}} = \bigoplus_i \pi_{\mathcal{Y}}^* T_i^{\oplus h_i(\mathcal{Y})}, \quad Ob_{\mathcal{B}} = \bigoplus_i T_i^{\oplus h_i(\mathcal{B})}.$$

The numbers $h_i(\mathcal{Y})$ and $h_i(\mathcal{B})$ are given by (5.1). It follows easily from the previous discussion that $h_i(\mathcal{Y}) = h_i(\mathcal{B})$. This proves (5.8).

Alternatively, we may use Construction 5.2 to prove (5.8) at the level of K -theory classes. The key point is that the class $S_i^{\mathcal{B}}$ is pulled back to the corresponding class $S_i^{\mathcal{Y}}$ under the natural map $I\mathcal{Y} \rightarrow I\mathcal{B}$ between inertia orbifolds. The details are left to the reader. \square

Proof of (5.9). Denote by $\widehat{f} : \widehat{\mathcal{Y}} \rightarrow \mathcal{B}$ the natural map. The map \widehat{f} is étale, and $T\widehat{\mathcal{Y}} = \widehat{f}^*T\mathcal{B}$. Consider the situation of Construction 5.1. Let $(x, [\rho], (q_1, q_2)) \in \widehat{\mathcal{Y}}_{(2)}$ and let $(x, (q_1, q_2)) \in \mathcal{B}_{(2)}$ be its image under $\pi_{\widehat{\mathcal{Y}}}$. Then we have a decomposition $\mathbf{p}_{\mathcal{B}_{(2)}}^*T\mathcal{B}|_{(x, (g, h))} = \bigoplus V_i \otimes T_i$ of vector bundles, where the V_i 's are irreducible representations of the group $\langle q_1, q_2 \rangle$ and the T_i 's are vector bundles over the component of $\mathcal{B}_{(2)}$ containing $(x, (q_1, q_2))$. Note that $T\widehat{\mathcal{Y}} = \widehat{f}^*T\mathcal{B}$ and $\mathbf{p}_{\mathcal{B}_{(2)}} \circ \pi_{\widehat{\mathcal{Y}}} = \widehat{f} \circ \mathbf{p}_{\widehat{\mathcal{Y}}_{(2)}}$. It follows that the decomposition of the bundle $\mathbf{p}_{\widehat{\mathcal{Y}}_{(2)}}^*T\widehat{\mathcal{Y}}$ as in Construction 5.1 is given by

$$\mathbf{p}_{\widehat{\mathcal{Y}}_{(2)}}^*T\widehat{\mathcal{Y}} = \bigoplus V_i \otimes \pi_{\widehat{\mathcal{Y}}}^*T_i.$$

According to Construction 5.1, we have

$$Ob_{\widehat{\mathcal{Y}}} = \bigoplus_i \pi_{\widehat{\mathcal{Y}}}^*T_i^{\oplus h_i(\widehat{\mathcal{Y}})}, \quad Ob_{\mathcal{B}} = \bigoplus_i T_i^{\oplus h_i(\mathcal{B})}.$$

The numbers $h_i(\widehat{\mathcal{Y}})$ and $h_i(\mathcal{B})$ are given by (5.1). It follows easily from the previous discussion that $h_i(\widehat{\mathcal{Y}}) = h_i(\mathcal{B})$. This proves (5.9).

Alternatively, we may use Construction 5.2 to prove (5.9) at the level of K -theory classes. The details are left to the reader. \square

Recall that locally the base \mathcal{B} can be presented as a quotient $[M/Q]$, the gerbe \mathcal{Y} can be presented as $[M/H]$, where the finite groups H, Q fit into an exact sequence

$$1 \rightarrow G \rightarrow H \rightarrow Q \rightarrow 1.$$

Over $[M/Q]$, the dual $\widehat{\mathcal{Y}}$ is presented as the quotient $[M \times \widehat{G}/Q]$. We define a function $M \times \widehat{G} \rightarrow \mathbb{Q}$ by $(x, [\rho]) \mapsto (\dim V_\rho / |G|)^2$. Since representations belonging to the same Q orbit all have the same dimension, this function descends to a function

$$w : \widehat{\mathcal{Y}} \rightarrow \mathbb{Q}.$$

Clearly, w only depends on connected components of $\widehat{\mathcal{Y}}$. Let $w(\widehat{\mathcal{Y}}_i)$ denote the value of w on the component $\widehat{\mathcal{Y}}_i$.

Theorem 5.8. *Let $\alpha_1, \alpha_2, \alpha_3 \in H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ be classes whose decompositions with respect to (5.5) are given by (5.6). Then we have*

$$(5.10) \quad \langle \Psi(\alpha_1), \Psi(\alpha_2), \Psi(\alpha_3) \rangle^{\mathcal{Y}} = \sum_i w(\widehat{\mathcal{Y}}_i) \langle \alpha_{1i}, \alpha_{2i}, \alpha_{3i} \rangle_{c_i}^{\widehat{\mathcal{Y}}_i}.$$

Proof. By the definitions of the symbols $\langle -, -, - \rangle^{\mathcal{Y}}$ and $\langle -, -, - \rangle_{c_i}^{\widehat{\mathcal{Y}}_i}$, we see that (5.10) can be written as

$$(5.11) \quad \begin{aligned} & \int_{\mathcal{Y}_{(2)}} ev_{\mathcal{Y},1}^* \Psi(\alpha_1) \cup ev_{\mathcal{Y},2}^* \Psi(\alpha_2) \cup ev_{\mathcal{Y},3}^* \Psi(\alpha_3) \cup e(Ob_{\mathcal{Y}}) \\ &= \sum_i w(\widehat{\mathcal{Y}}_i) \int_{\widehat{\mathcal{Y}}_{i(2)}} ev_{\widehat{\mathcal{Y}}_i,1}^* \alpha_{1i} \cup ev_{\widehat{\mathcal{Y}}_i,2}^* \alpha_{2i} \cup ev_{\widehat{\mathcal{Y}}_i,3}^* \alpha_{3i} \cup e(Ob_{\widehat{\mathcal{Y}}_i}). \end{aligned}$$

Let $\pi_{\widehat{\mathcal{Y}}_i} : \widehat{\mathcal{Y}}_{i(2)} \rightarrow \mathcal{B}_{(2)}$ denote the natural map induced by the map $\widehat{\mathcal{Y}}_i \rightarrow \mathcal{B}$. Then (5.9) implies that $Ob_{\widehat{\mathcal{Y}}_i} = \pi_{\widehat{\mathcal{Y}}_i}^* Ob_{\mathcal{B}}$. Together with (5.8), it implies that (5.11) can be rewritten as

$$(5.12) \quad \begin{aligned} & \int_{\mathcal{B}_{(2)}} \pi_{\mathcal{Y}*} (ev_{\mathcal{Y},1}^* \Psi(\alpha_1) \cup ev_{\mathcal{Y},2}^* \Psi(\alpha_2) \cup ev_{\mathcal{Y},3}^* \Psi(\alpha_3)) \cup e(Ob_{\mathcal{B}}) \\ &= \sum_i w(\widehat{\mathcal{Y}}_i) \int_{\mathcal{B}_{(2)}} \pi_{\widehat{\mathcal{Y}}_i*} (ev_{\widehat{\mathcal{Y}}_i,1}^* \alpha_{1i} \cup ev_{\widehat{\mathcal{Y}}_i,2}^* \alpha_{2i} \cup ev_{\widehat{\mathcal{Y}}_i,3}^* \alpha_{3i}) \cup e(Ob_{\mathcal{B}}). \end{aligned}$$

Thus (5.10) follows from the following equality of classes in $H^\bullet(\mathcal{B}_{(2)}, \mathbb{C})$:

$$(5.13) \quad \begin{aligned} & \pi_{\mathcal{Y}*} (ev_{\mathcal{Y},1}^* \Psi(\alpha_1) \cup ev_{\mathcal{Y},2}^* \Psi(\alpha_2) \cup ev_{\mathcal{Y},3}^* \Psi(\alpha_3)) \\ &= \sum_i w(\widehat{\mathcal{Y}}_i) \pi_{\widehat{\mathcal{Y}}_i*} (ev_{\widehat{\mathcal{Y}}_i,1}^* \alpha_{1i} \cup ev_{\widehat{\mathcal{Y}}_i,2}^* \alpha_{2i} \cup ev_{\widehat{\mathcal{Y}}_i,3}^* \alpha_{3i}). \end{aligned}$$

The proof (5.13) is a little technical and lengthy, and will be given in Section 5.3. \square

Corollary 5.9. *Let $\alpha_1, \alpha_2 \in H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ be classes whose decompositions with respect to (5.5) are given by (5.6). Then*

$$(5.14) \quad (\Psi(\alpha_1), \Psi(\beta_1))_{orb}^{\mathcal{Y}} = \sum_i w(\widehat{\mathcal{Y}}_i) (\alpha_{1i}, \alpha_{2i})_{orb, c_i}^{\widehat{\mathcal{Y}}_i}.$$

Proof. Let $\alpha_3 = \oplus_i 1_i \in \oplus_i H^0(\widehat{\mathcal{Y}}_i, c_i, \mathbb{C})$, where $1_i \in H^0(\widehat{\mathcal{Y}}_i, c_i, \mathbb{C})$ is the identity element with respect to the product \star_{c_i} . Then by the description of $I = \Psi^{-1}$ in (4.14), we have that $\Psi(\alpha_3) = 1 \in H^0(\mathcal{Y}, \mathbb{C})$ is the class Poincaré dual to the fundamental class, and the corollary follows from (5.10) since $\langle \Psi(\alpha_1), \Psi(\alpha_2), 1 \rangle^{\mathcal{Y}} = (\Psi(\alpha_1), \Psi(\beta_1))_{orb}^{\mathcal{Y}}$ and $\langle \alpha_{1i}, \alpha_{2i}, 1_i \rangle_{c_i}^{\widehat{\mathcal{Y}}_i} = (\alpha_{1i}, \alpha_{2i})_{orb, c_i}^{\widehat{\mathcal{Y}}_i}$. \square

Theorem 5.10. *The map $\Psi : H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C}) \rightarrow H_{CR}^\bullet(\mathcal{Y}, \mathbb{C})$ is an isomorphism of rings.*

Proof. Since Ψ is an additive isomorphism, it suffices to prove that for $\alpha_1, \alpha_2 \in H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$, we have

$$\Psi(\alpha_1) \star_{orb} \Psi(\alpha_2) = \Psi(\alpha_1 \star_c \alpha_2).$$

By the non-degeneracy of the pairing $(-, -)_{orb}^{\mathcal{Y}}$, this is equivalent to

$$(5.15) \quad (\Psi(\alpha_1) \star_{orb} \Psi(\alpha_2), \Psi(\alpha_3))_{orb}^{\mathcal{Y}} = (\Psi(\alpha_1 \star_c \alpha_2), \Psi(\alpha_3))_{orb}^{\mathcal{Y}}, \quad \text{for any } \alpha_3 \in H_{orb}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C}).$$

The left-hand side of (5.15) is $\langle \Psi(\alpha_1), \Psi(\alpha_2), \Psi(\alpha_3) \rangle^{\mathcal{Y}}$. Suppose that the classes $\alpha_1, \alpha_2, \alpha_3$ are decomposed as in (5.6). Then by (5.14), the right-hand side of (5.15) is equal to

$$\begin{aligned} & (\Psi(\alpha_1 \star_c \alpha_2), \Psi(\alpha_3))_{orb}^{\mathcal{Y}} \\ &= \sum_i w(\widehat{\mathcal{Y}}_i) (\alpha_{1i} \star_{c_i} \alpha_{2i}, \alpha_{3i})_{orb, c_i}^{\widehat{\mathcal{Y}}_i} \\ &= \sum_i w(\widehat{\mathcal{Y}}_i) \langle \alpha_{1i}, \alpha_{2i}, \alpha_{3i} \rangle_{c_i}^{\widehat{\mathcal{Y}}_i}. \end{aligned}$$

Thus, (5.15) is equivalent to (5.10). The theorem follows. \square

5.3. Proof of (5.13). Recall that in the proof of Theorem 4.17, we constructed an explicit formula (4.15)

$$I(\alpha)([\rho], q) = \sum_g \frac{1}{\dim(V_\rho)} \alpha(g, q) \operatorname{tr}(\rho(g) T_q^{[\rho]-1})$$

of the isomorphism $I : H_{\text{CR}}^\bullet(\mathcal{Y}, \mathbb{C}) \rightarrow H_{\text{orb}}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$. In particular, a cohomology class of $H_{\text{CR}}^\bullet(\mathcal{Y}, \mathbb{C})$ can be represented by a \mathfrak{H} -invariant differential form on $\mathfrak{H}^{(0)} := \{(g, q) \in \mathfrak{H} \mid s(q) = t(q)\}$, and a cohomology class in $H_{\text{orb}}^\bullet(\widehat{\mathcal{Y}}, c, \mathbb{C})$ can be represented by a differential form supported on $\mathfrak{Q}' := \{([\rho], q) : q([\rho]) = [\rho], s(q) = t(q)\} \subset \widehat{G} \rtimes_c \mathfrak{Q}$ that is invariant under the c -twisted conjugation action by $\widehat{G} \rtimes_c \mathfrak{Q}$. To prove equation (5.13), we proceed by showing that the isomorphism I , which is the inverse of the map Ψ , satisfies equation (5.13). We need the following lemmas.

Lemma 5.11. *At $(g, q) \in \mathfrak{H}^{(0)}$, fix a basis $\{\omega_j\}$ of $\wedge^\bullet T_{g,q}^*(\mathfrak{H}^{(0)})((\hbar))$. Write $\alpha(g, q) = \sum_\sigma \alpha(g, q)^j \omega_j$. Then $\sum_g \alpha(g, q)^j$ satisfies*

$$\sum_g \alpha(g, q)^j g_0 g = \sum_g \alpha(g, q)^j g \operatorname{Ad}_{\sigma(q)}(g_0),$$

for any $g_0 \in G$.

Proof. Since $\alpha = \sum_{g,q} \alpha(g, q)$ is invariant under the conjugation action of H , for any $g_0 \in G$, we have

$$g_0 \sum_{g,q} \alpha(g, q)(g, q) = \sum_{g,q} \alpha(g, q)(g, q) g_0.$$

Since G acts on \mathfrak{H}_0 trivially, this implies that

$$\sum_{g,q} \alpha(g, q)(g_0 g, q) = \sum_{g,q} \alpha(g, q)(g \operatorname{Ad}_{\sigma(q)}(g_0), q).$$

We deduce the lemma from the above equation by looking at every q component. \square

Lemma 5.12. *Let α be a closed differential form on $\mathfrak{H}^{(0)} = \{(g, q) \in \mathfrak{H} : s(q) = t(q)\} \subset \mathfrak{H}$ that is invariant under the conjugation action of \mathfrak{H} on $\mathfrak{H}^{(0)}$. Following the same conventions as in Lemma 5.11, we write $\alpha = \sum_{j,g,q} \alpha(g, q)^j$. Then*

- (1) *if $q([\rho]) = [\rho]$ and ρ is an irreducible representation of G , then $\sum_g \alpha(g, q)^\sigma \rho(g) T_q^{[\rho]-1}$ is a scalar multiple of the identity operator.*
- (2) *if $q([\rho]) \neq [\rho]$, then $\sum_g \alpha(g, q)^\sigma \rho(g) = 0$.*

Proof. When $q([\rho]) = [\rho]$, we apply ρ to $\sum_g \alpha(g, q)^j g$. By Lemma 5.11, we have

$$\begin{aligned} \rho \left(\sum_g \alpha(g, q)^j g_0 g \right) &= \rho \left(\sum_g \alpha(g, q)^j g \operatorname{Ad}_{\sigma(q)}(g_0) \right), \\ \rho(g_0) \rho \left(\sum_g \alpha(g, q)^j g \right) &= \rho \left(\sum_g \alpha(g, q)^j g \right) \rho(\operatorname{Ad}_{\sigma(q)}(g_0)) \\ &= \rho \left(\sum_g \alpha(g, q)^j g \right) (T_q^{[\rho]-1} q([\rho])(g_0) T_q^{[\rho]}), \end{aligned}$$

where in the last equality, we used the property of $\rho(\text{Ad}_{\sigma(q)}(g))$ in Section 3.2. As $q([\rho]) = [\rho]$, we have that

$$\rho(g_0)\rho\left(\sum_g \alpha(g, q)^j g\right) T_q^{[\rho]^{-1}} = \rho\left(\sum_g \alpha(g, q)^j g\right) T_q^{[\rho]^{-1}} \rho(g_0).$$

If ρ is an irreducible representation of G , then the above equation together with Schur's lemma imply that $\rho\left(\sum_g \alpha(g, q)^j g\right) T_q^{[\rho]^{-1}}$ is a scalar multiple of the identity operator. This proves the first claim.

For the second claim, we consider the following equality from Lemma 5.11:

$$\sum_g \alpha(g, q)(g_0 g, q) = \sum_g \alpha(g, q)(g \text{Ad}_{\sigma(q)}(g_0), q).$$

This is equivalent to

$$\sum_g \alpha(g, q)g = \sum_g \alpha(g, q)(g_0^{-1}g \text{Ad}_{\sigma(q)}(g_0), q).$$

We apply ρ to both sides of the above equation and obtain

$$\begin{aligned} \sum_g \alpha(g, q)\rho(g) &= \sum_g \alpha(g, q)\rho(g_0^{-1})\rho(g \text{Ad}_{\sigma(q)}(g_0)g^{-1})\rho(g) \\ &= \frac{1}{|G|} \sum_g \alpha(g, q) \sum_{g_0} \rho(g_0^{-1})\rho(g \text{Ad}_{\sigma(q)}(g_0)g^{-1})\rho(g). \end{aligned}$$

We observe that $\rho(g \text{Ad}_{\sigma(q)}(-)g^{-1})$ is an irreducible representation of G equivalent to $\rho(\text{Ad}_{\sigma(q)}(-))$, which is equivalent to $q([\rho])$.

If $q([\rho]) \neq [\rho]$, then by [31, Theorem 4.2],

$$\sum_g \alpha(g, q)\rho(g) = \frac{1}{|G|} \sum_g \alpha(g, q) \sum_{g_0} \rho(g_0^{-1})\rho(g \text{Ad}_{\sigma(q)}(g_0)g^{-1})\rho(g) = 0.$$

□

Consider $\alpha_1, \alpha_2, \alpha_3 \in H^\bullet(I\mathcal{Y}, \mathbb{C}((\hbar)))$. We represent the α_i 's as \mathfrak{H} invariant differential forms over $\mathfrak{H}^{(0)}$. Now we compute

$$\pi_{\mathcal{Y}*} \left(ev_{\mathcal{Y},1}^*(\alpha_1) \cup ev_{\mathcal{Y},2}^*(\alpha_2) \cup ev_{\mathcal{Y},3}^*(\alpha_3) \right).$$

For $q_1, q_2 \in \mathfrak{Q}^{(0)} = \{q \in \mathfrak{Q} : s(q) = t(q)\}$ and $q_3 = (q_1 q_2)^{-1}$, it is equal to

$$(5.16) \quad \frac{1}{|G|} \sum_{\substack{g_1, g_2, g_3 \\ (g_1, q_1)(g_2, q_2)(g_3, q_3) = 1}} \alpha_1(g_1, q_1) q_1^*(\alpha_2(g_2, q_2)) q_1^* q_2^*(\alpha_3(g_3, q_3)).$$

Now we compute

$$\sum_{\mathbf{i}} w(\widehat{\mathcal{Y}}_{\mathbf{i}}) \pi_{\widehat{\mathcal{Y}}_{\mathbf{i}}^*} \left(ev_{\widehat{\mathcal{Y}}_{\mathbf{i}},1}^* I(\alpha_{1\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i}},2}^* I(\alpha_{2\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i}},3}^* I(\alpha_{3\mathbf{i}}) \right).$$

Using equation (4.14), for $q_1, q_2, q_3 = (q_1 q_2)^{-1} \in \mathfrak{Q}^{(0)}$, it is equal to

$$\begin{aligned}
& \sum_{\rho, q_1([\rho])=q_2([\rho])=\rho} \frac{\dim(V_\rho)^2}{|G|^2} \frac{c^{[\rho]}(q_1, q_2)c^{[\rho]}(q_1 q_2, q_3)}{\dim(V_\rho)^3} \sum_{g_1} \alpha_1(g_1, q_1) \operatorname{tr}(\rho(g_1)T_{q_1}^{[\rho]-1}) \\
& \quad \sum_{g_2} q_1^*(\alpha_2(g_2, q_2)) \operatorname{tr}(\rho(g_2)T_{q_2}^{[\rho]-1}) \sum_{g_3} q_1^* q_2^*(\alpha_3(g_3, q_3)) \operatorname{tr}(\rho(g_3)T_{q_3}^{[\rho]-1}) \\
&= \frac{1}{|G|} \sum_{\rho, q_1([\rho])=q_2([\rho])=\rho} \frac{c^{[\rho]}(q_1, q_2)c^{[\rho]}(q_1 q_2, q_3)}{\dim(V_\rho)|G|} \sum_{g_1, g_2, g_3} \alpha_1(g_1, q_1) \operatorname{tr}(\rho(g_1)T_{q_1}^{[\rho]-1}) \\
& \quad q_1^*(\alpha_2(g_2, q_2)) \operatorname{tr}(\rho(g_2)T_{q_2}^{[\rho]-1}) q_1^* q_2^*(\alpha_3(g_3, q_3)) \operatorname{tr}(\rho(g_3)T_{q_3}^{[\rho]-1}).
\end{aligned}$$

Using the fact that $\sum_{g_i} \alpha_i(g_i, q_i) \rho(g_i) T_{q_i}^{[\rho]-1}$ is a scalar operator, we can rewrite the term

$$\sum_{g_1, g_2, g_3} \alpha_1(g_1, q_1) \operatorname{tr}(\rho(g_1)T_{q_1}^{[\rho]-1}) q_1^*(\alpha_2(g_2, q_2)) \operatorname{tr}(\rho(g_2)T_{q_2}^{[\rho]-1}) q_1^* q_2^*(\alpha_3(g_3, q_3)) \operatorname{tr}(\rho(g_3)T_{q_3}^{[\rho]-1})$$

in the above equation as

$$\sum_{g_1, g_2, g_3} \dim(V_\rho)^2 \alpha_1(g_1, q_1) q_1^*(\alpha_2(g_2, q_2)) q_1^* q_2^*(\alpha_3(g_3, q_3)) \operatorname{tr}(\rho(g_1)T_{q_1}^{[\rho]-1} \rho(g_2)T_{q_2}^{[\rho]-1} \rho(g_3)T_{q_3}^{[\rho]-1}).$$

The operator $\rho(g_1)T_{q_1}^{[\rho]-1} \rho(g_2)T_{q_2}^{[\rho]-1} \rho(g_3)T_{q_3}^{[\rho]-1}$ can be written as

$$\rho(g_1)T_{q_1}^{[\rho]-1} \rho(g_2)T_{q_1}^{[\rho]} T_{q_1}^{[\rho]-1} T_{q_2}^{[\rho]-1} \rho(g_3)T_{q_1 q_2}^{[\rho]} T_{q_1 q_2}^{[\rho]-1} T_{q_3}^{[\rho]-1}.$$

Using the defining equation (3.5) of $c^{[\rho]}(q_1, q_2)$ and $c^{[\rho]}(q_1 q_2, q_3)$, we have

$$\begin{aligned}
c^{[\rho]}(q_1, q_2) T_{q_1}^{[\rho]-1} T_{q_2}^{[\rho]-1} &= \rho(\tau(q_1, q_2)) T_{q_1 q_2}^{[\rho]-1}, \\
c^{[\rho]}(q_1 q_2, q_3) T_{q_1 q_2}^{[\rho]-1} T_{q_3}^{[\rho]-1} &= \rho(\tau(q_1 q_2, q_3)) T_{q_1 q_2 q_3}^{[\rho]-1}.
\end{aligned}$$

With the above considerations, $\sum_{\mathbf{i}} w(\widehat{\mathcal{Y}}_{\mathbf{i}}) \pi_{\widehat{\mathcal{Y}}_{\mathbf{i}}^*} \left(ev_{\widehat{\mathcal{Y}}_{\mathbf{i},1}}^* I(\alpha_{1\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i},2}}^* I(\alpha_{2\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i},3}}^* I(\alpha_{3\mathbf{i}}) \right)$ can be written as

$$\begin{aligned}
& \frac{1}{|G|^2} \sum_{\substack{\rho, q_1([\rho])=q_2([\rho])=\rho, \\ g_1, g_2, g_3}} \dim(V_\rho) \alpha_1(g_1, q_1) q_1^* \alpha_2(g_2, q_2) q_1^* q_2^*(\alpha_3(g_3, q_3)) \\
& \quad \operatorname{tr} \left(\rho(g_1)T_{q_1}^{[\rho]-1} \rho(g_2)T_{q_1}^{[\rho]} \rho(\tau(q_1, q_2))T_{q_1 q_2}^{[\rho]-1} \rho(g_3)T_{q_1 q_2}^{[\rho]} \rho(\tau(q_1 q_2, q_3)) \right) \\
&= \frac{1}{|G|^2} \sum_{\substack{\rho, q_1([\rho])=q_2([\rho])=\rho, \\ g_1, g_2, g_3}} \dim(V_\rho) \alpha_1(g_1, q_1) q_1^* \alpha_2(g_2, q_2) q_1^* q_2^*(\alpha_3(g_3, q_3)) \\
& \quad \operatorname{tr} \left(\rho(g_1) \operatorname{Ad}_{\sigma(q_1)}(g_2) \tau(q_1, q_2) \operatorname{Ad}_{\sigma(q_1 q_2)}(g_3) \tau(q_1 q_2, q_3) \right).
\end{aligned}$$

By Lemma 5.12, (2), we know that the restriction $q_1([\rho]) = q_2([\rho]) = [\rho]$ in the above summation can be dropped since, otherwise, the contribution vanishes. In summary, we have

$$\begin{aligned}
& \sum_{\mathbf{i}} w(\widehat{\mathcal{Y}}_{\mathbf{i}}) \pi_{\widehat{\mathcal{Y}}_{\mathbf{i}}^*} \left(ev_{\widehat{\mathcal{Y}}_{\mathbf{i},1}}^* I(\alpha_{1\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i},2}}^* I(\alpha_{2\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i},3}}^* I(\alpha_{3\mathbf{i}}) \right) \\
&= \frac{1}{|G|^2} \sum_{\rho} \sum_{g_1, g_2, g_3} \dim(V_{\rho}) \alpha_1(g_1, q_1) q_1^* \alpha_2(g_2, q_2) q_1^* q_2^* (\alpha_3(g_3, q_3)) \\
&\quad \text{tr} \left(\rho(g_1 \text{Ad}_{\sigma(q_1)}(g_2) \tau(q_1, q_2) \text{Ad}_{\sigma(q_1 q_2)}(g_3) \tau(q_1 q_2, q_3)) \right) \\
&= \sum_{g_1, g_2, g_3} \alpha_1(g_1, g_1) q_1^* (\alpha_2(g_2, q_2)) q_1^* q_2^* (\alpha_3(g_3, q_3)) \\
&\quad \frac{1}{|G|^2} \sum_{\rho} \dim(V_{\rho}) \text{tr} \left(\rho(g_1 \text{Ad}_{\sigma(q_1)}(g_2) \tau(q_1, q_2) \text{Ad}_{\sigma(q_1 q_2)}(g_3) \tau(q_1 q_2, q_3)) \right).
\end{aligned}$$

By the orthogonality relations of characters of G (see, e.g., [32, (2.20)]), we know that the above sum vanishes unless $g_1 \text{Ad}_{\sigma(q_1)}(g_2) \tau(q_1, q_2) \text{Ad}_{\sigma(q_1 q_2)}(g_3) \tau(q_1 q_2, q_3) = 1$.

We conclude that

$$\begin{aligned}
& \sum_{\mathbf{i}} w(\widehat{\mathcal{Y}}_{\mathbf{i}}) \pi_{\widehat{\mathcal{Y}}_{\mathbf{i}}^*} \left(ev_{\widehat{\mathcal{Y}}_{\mathbf{i},1}}^* I(\alpha_{1\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i},2}}^* I(\alpha_{2\mathbf{i}}) \cup ev_{\widehat{\mathcal{Y}}_{\mathbf{i},3}}^* I(\alpha_{3\mathbf{i}}) \right) \\
&= \frac{1}{|G|} \sum_{(g_1, q_1)(g_2, q_2)(g_3, q_3)=1} \alpha_1(g_1, g_1) q_1^* (\alpha_2(g_2, q_2)) q_1^* q_2^* (\alpha_3(g_3, q_3)),
\end{aligned}$$

which is exactly the expression (5.16) for

$$\pi_{\mathcal{Y}^*} \left(ev_{\mathcal{Y},1}^* (\alpha_1) \cup ev_{\mathcal{Y},2}^* (\alpha_2) \cup ev_{\mathcal{Y},3}^* (\alpha_3) \right).$$

By Proposition 4.16, we know that the map I defined by equation (4.14) is an isomorphism between $H_{CR}^{\bullet}(\mathcal{Y})((\hbar))$ and $H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C}((\hbar)))$. Since I is independent of \hbar , I restricts to a linear map from $H_{CR}^{\bullet}(\mathcal{Y}, \mathbb{C})$ to $H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C})$. This restriction map must be an isomorphism as it induces an isomorphism from $H_{CR}^{\bullet}(\mathcal{Y})((\hbar))$ to $H_{orb}^{\bullet}(\widehat{\mathcal{Y}}, c, \mathbb{C}((\hbar)))$. Therefore, we have proved equation (5.13) for the cohomology group $H^{\bullet}(\mathcal{B}_{(2)}, \mathbb{C})$.

6. GROMOV-WITTEN THEORY OF BH

In this section, we discuss how the results in Section 3.3 can be used to deduce Gromov-Witten theoretic consequences for the G -gerbe BH over BQ , given by the exact sequence (3.1). The Gromov-Witten theory of the classifying orbifold BH of a finite group has been studied in great detail in [38], to which we refer the readers for basic definitions and discussions.

6.1. Quantum Cohomology. The orbifold quantum cohomology ring $QH_{orb}^{\bullet}(\mathcal{X})$ of a compact symplectic orbifold \mathcal{X} is a deformation of the cohomology $H^{\bullet}(I\mathcal{X}, \mathbb{C})$ of the inertia orbifold $I\mathcal{X}$ constructed using genus 0 Gromov-Witten invariants of \mathcal{X} . Details of the construction can be found in [22] (see [2] for the construction in the algebro-geometric context). In the special case of BH , detail discussions on $QH_{orb}^{\bullet}(BH)$ can be found in [38].

It is known (see [23] and [38]) that the orbifold quantum cohomology ring of BH is simply the center of the group ring $\mathbb{C}H$, i.e.,

$$(6.1) \quad QH_{orb}^{\bullet}(BH) \simeq Z(\mathbb{C}H).$$

Consider the Q -action on \widehat{G} as discussed in Section 3.1. The set \widehat{G} may be divided into a disjoint union of Q -orbits,

$$\widehat{G} = \cup_{O_i \in \text{Orb}^Q(\widehat{G})} O_i.$$

For each O_i , pick $[\rho_i] \in O_i$ and let $Q_i = \text{Stab}([\rho_i]) \subset Q$ denote the stabilizer subgroup of $[\rho_i]$. Then as orbifolds, we have

$$[\widehat{G}/Q] \simeq \cup_i BQ_i.$$

Each BQ_i admits a flat $U(1)$ -gerbe c_i obtained from the flat $U(1)$ -gerbe c on $\widehat{BH} = [\widehat{G}/Q]$. By Theorem 3.3 we have

Proposition 6.1. *The twisted groupoid algebra $C(\widehat{G} \rtimes Q, c)$ is Morita equivalent to the direct sum $\oplus_i C(Q_i, c_i)$ of twisted group algebras.*

As reviewed in Section 5.1, given a compact symplectic orbifold \mathcal{X} and a flat $U(1)$ -gerbe c on \mathcal{X} , one can consider the cohomology $H^\bullet(I\mathcal{X}, \mathcal{L}_c)$ with coefficients in the inner local system $\mathcal{L}_c \rightarrow I\mathcal{X}$ associated with c . The work [55] constructs a deformation, $QH_{orb,c}^\bullet(\mathcal{X})$, of $H^\bullet(I\mathcal{X}, \mathcal{L}_c)$ using *gerbe-twisted Gromov-Witten invariants* of \mathcal{X} . The basics of gerbe-twisted Gromov-Witten invariants in the case of (BQ_i, c_i) will be reviewed below. The construction in the general case can be found in [55].

It is known (see [59, Example 6.4] and [55]) that the c_i -twisted orbifold quantum cohomology of BQ_i coincides with the center of the twisted group algebra, i.e.,

$$(6.2) \quad QH_{orb,c_i}^\bullet(BQ_i) \simeq Z(C(Q_i, c_i)).$$

We thus obtain

$$(6.3) \quad QH_{orb}^\bullet(BH) \stackrel{(6.1)}{\simeq} Z(\mathbb{C}H) \stackrel{I}{\simeq} \oplus_i Z(C(Q_i, c_i)) \stackrel{(6.2)}{\simeq} \oplus_i QH_{orb,c_i}^\bullet(BQ_i) = QH_{orb,c}^\bullet([\widehat{G}/Q]),$$

where the map I is defined in (3.10).

As stated in [38, Corollary 3.3], the map (6.1) identifies the orbifold Poincaré pairing $(-, -)_{orb}^{BH}$ on BH with the pairing defined by the trace tr_H . Similarly, one can deduce from [59] that the map (6.2) identifies the orbifold Poincaré pairing on $QH_{orb,c}^\bullet(BQ_i)$ with the one defined by the trace $\text{tr}_{[\rho_i]}$. Now Proposition 3.5 implies that

Lemma 6.2. *The isomorphism in (6.3) identifies the orbifold Poincaré pairing $(-, -)_{orb}^{BH}$ with the following rescaled orbifold Poincaré pairing:*

$$\bigoplus_i \left(\frac{\dim V_{\rho_i}}{|G|} \right)^2 (-, -)_{orb}^{BQ_i},$$

where $\dim V_{\rho_i}$ is the dimension of the irreducible G -representation $\rho_i : G \rightarrow \text{End}(V_{\rho_i})$.

6.2. Gromov-Witten invariants. We may view (6.3) as a decomposition of $QH_{orb}^\bullet(BH)$ into a direct sum. In this section, we discuss how to extend this decomposition to the full Gromov-Witten theory. In this subsection, the letters g and g_i denote genera of curves.

We begin with a brief review of the gerbe-twisted orbifold Gromov-Witten theory for (BQ_i, c_i) . Details can be found in [59] and [55]. Let

$$IBQ_i = \bigcup_{\langle q \rangle \subset Q_i} BC_{Q_i}(q)$$

be the decomposition of the inertia orbifold of BQ_i , where the union is taken over the conjugacy classes $\langle q \rangle$ of Q_i , and $C_{Q_i}(q) \subset Q_i$ is the centralizer subgroup of $q \in Q$. Let

$$\mathcal{L}_{c_i} \rightarrow IBQ_i$$

be the inner local system associated with the $U(1)$ -gerbe c_i (see [59] and [55] for its construction). Given classes $\alpha_j \in H^\bullet(BC_{Q_i}(q_j), \mathcal{L}_{c_i})$, $1 \leq j \leq n$ and non-negative integer a_1, \dots, a_n , by the construction of gerbe-twisted Gromov-Witten invariants, there exists an isomorphism of line bundles (see [55], Section 5.2)

$$\theta_{\langle q_1 \rangle, \dots, \langle q_n \rangle} : \otimes_{j=1}^n (\mathcal{L}_{c_i}|_{BC_{Q_i}(q_j)}) \rightarrow \mathbb{C},$$

such that the invariant is defined to be (see [55], Definition 5.4)

$$(6.4) \quad \langle \tau_{a_1}(\alpha_1), \dots, \tau_{a_n}(\alpha_n) \rangle_{g,n}^{BQ_i, c} = \int_{\overline{\mathcal{M}}_{g,n}(BQ_i, \langle q_1 \rangle, \dots, \langle q_n \rangle)} (\theta_{\langle q_1 \rangle, \dots, \langle q_n \rangle})_* \left(\prod_{j=1}^n ev_j^* \alpha_j \right) \prod_{j=1}^n \pi^* \psi_j^{a_j}.$$

Here, $\theta(\alpha_1, \dots, \alpha_n) := (\theta_{\langle q_1 \rangle, \dots, \langle q_n \rangle})_* \left(\prod_{j=1}^n ev_j^* \alpha_j \right) \in \mathbb{C}$. The integral is taken over the moduli space

$$\overline{\mathcal{M}}_{g,n}(BQ_i, \langle q_1 \rangle, \dots, \langle q_n \rangle)$$

of n -pointed genus g orbifold stable maps to BQ_i such that the orbifold structures at marked points are determined by the conjugacy classes $\langle q_1 \rangle, \dots, \langle q_n \rangle$ of Q_i . In the above equation, we denote by

$$\pi : \overline{\mathcal{M}}_{g,n}(BQ_i, \langle q_1 \rangle, \dots, \langle q_n \rangle) \rightarrow \overline{\mathcal{M}}_{g,n}$$

the natural map obtained by forgetting the orbifold structures and by $\psi_j \in H^2(\overline{\mathcal{M}}_{g,n})$ the descendant classes. We refer the reader to [38] for a detailed discussion of orbifold stable maps to BQ_i and their moduli spaces.

Let

$$\langle \tau_{a_1}, \dots, \tau_{a_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \psi_j^{a_j}$$

be the descendant integral over $\overline{\mathcal{M}}_{g,n}$. The projection formula implies

$$(6.5) \quad \begin{aligned} \langle \tau_{a_1}(\alpha_1), \dots, \tau_{a_n}(\alpha_n) \rangle_{g,n}^{BQ_i, c} &= (\deg \pi) \theta(\alpha_1, \dots, \alpha_n) \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \psi_j^{a_j} \\ &= (\deg \pi) \theta(\alpha_1, \dots, \alpha_n) \langle \tau_{a_1}, \dots, \tau_{a_n} \rangle_{g,n}. \end{aligned}$$

This discussion is similar to Proposition 3.4 of [38].

According to the proof of [38, Proposition 3.4], the degree $\deg \pi$ is equal to

$$(6.6) \quad \begin{aligned} &\Omega_g^{Q_i}(\langle q_1 \rangle, \dots, \langle q_n \rangle) \\ &:= \frac{1}{|Q_i|} \# \left\{ \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \sigma_1, \dots, \sigma_n \mid \prod_{j=1}^g [\alpha_j, \beta_j] = \prod_{k=1}^n \sigma_k, \sigma_k \in \langle q_k \rangle \text{ for all } k \right\}. \end{aligned}$$

Recall that, according to [59, Example 6.4], the cohomology vector space $H^\bullet(BC_{Q_i}(q), \mathcal{L}_{c_i})$ is 1-dimensional if $\langle q \rangle$ is a c_i -regular conjugacy class of Q_i and 0-dimensional otherwise. Recall that a conjugacy class $\langle q \rangle$ of Q_i is c_i -regular if $c_i(q_1, q)c_i(q, q_1)^{-1} = 1$ for every $q_1 \in C_{Q_i}(q)$. For a c_i -regular conjugacy class $\langle q \rangle$, we denote by $e_{\langle q \rangle}$ a generator of $H^\bullet(BC_{Q_i}(q), \mathcal{L}_{c_i})$.

Proposition 6.3. *The assignment $\Lambda_{g,n}^{BQ_i, c_i} : H_{orb, c_i}^\bullet(BQ_i)^{\otimes n} \rightarrow \mathbb{C}$ given by*

$$\alpha_1 \otimes \dots \otimes \alpha_n \mapsto \Lambda_{g,n}^{BQ_i, c_i}(\alpha_1, \dots, \alpha_n) := \Omega_g^{Q_i}(\langle q_1 \rangle, \dots, \langle q_n \rangle) \theta(\alpha_1, \dots, \alpha_n)$$

satisfies the following properties:

(1) (Forgetting tails)

$$(6.7) \quad \Lambda_{g,n}^{BQ_i, c_i}(e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}) = \Lambda_{g,n+1}^{BQ_i, c_i}(e_{\langle 1 \rangle}, e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}).$$

(2) (*Cutting loops*)

$$(6.8) \quad \Lambda_{g,n}^{BQ_i, c_i}(e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}) = \sum_{\langle q \rangle} |C_{Q_i}(q)| \Lambda_{g-1, n+2}^{BQ_i, c_i}(e_{\langle q \rangle}, e_{\langle q^{-1} \rangle}, e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}).$$

(3) (*Cutting trees*) For $g = g_1 + g_2$ and $\{1, \dots, n\} = P_1 \amalg P_2$, we have

$$(6.9) \quad \Lambda_{g,n}^{BQ_i, c_i}(e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}) = \sum_{\langle q \rangle} |C_{Q_i}(q)| \Lambda_{g_1, |P_1|+1}^{BQ_i, c_i}(\{e_{\langle q_i \rangle}\}_{i \in P_1}, e_{\langle q \rangle}) \Lambda_{g_2, |P_2|+1}^{BQ_i, c_i}(e_{\langle q^{-1} \rangle}, \{e_{\langle q_i \rangle}\}_{i \in P_2}).$$

Remark 6.4.

- (1) It is easy to see that $\Lambda_{g,n}^{BQ_i, c_i}(e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle})$ is independent of the ordering of conjugacy classes $\langle q_1 \rangle, \dots, \langle q_n \rangle$.
- (2) Note that the collection of maps $\{\Lambda_{g,n}^{BQ_i, c_i}\}$ are determined by their values on $e_{\langle q_1 \rangle} \otimes \dots \otimes e_{\langle q_n \rangle}$, and the proposition shows that the collection of gerbe-twisted Gromov-Witten invariants of BQ_i form a cohomological field theory¹⁵ (see, for example, [48, Chapter III] for a comprehensive introduction to cohomological field theory).

Proof. The proof of this proposition amounts to a repeat of the arguments in the proof of [38, Lemma 3.5]. Handling the factor $\theta(\alpha_1, \dots, \alpha_n)$ requires some properties of the inner local system (see [59]).

To prove the *forgetting tails* property (6.7), first note that for the inner local system \mathcal{L}_{c_i} , the restriction $\mathcal{L}_{c_i}|_{BC_{Q_i}(1)}$ is a trivial line bundle. Thus, $\theta_{\langle 1 \rangle, \langle q_1 \rangle, \dots, \langle q_n \rangle} = id \otimes \theta_{\langle q_1 \rangle, \dots, \langle q_n \rangle}$ and $\theta(e_{\langle 1 \rangle}, e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}) = \theta(e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle})$. This, together with the relevant property of $\Omega_g^{Q_i}$ (see [38], Proposition 3.5 (3)), implies (6.7).

To prove the *cutting loop* property (6.8), first note that by [38, Proposition 3.5 (2)], we have

$$\Omega_g^{Q_i}(\langle q_1 \rangle, \dots, \langle q_n \rangle) = \sum_{\langle q \rangle} |C_{Q_i}(q)| \Omega_{g-1}^{Q_i}(\langle q \rangle, \langle q^{-1} \rangle, \langle q_1 \rangle, \dots, \langle q_n \rangle).$$

Thus, it suffices to prove

$$(6.10) \quad \theta(e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}) = \theta(e_{\langle q \rangle}, e_{\langle q^{-1} \rangle}, e_{\langle q_1 \rangle}, \dots, e_{\langle q_n \rangle}).$$

This follows from the following two facts:

- (1) $\theta_{\langle q \rangle, \langle q^{-1} \rangle, \langle q_1 \rangle, \dots, \langle q_n \rangle} = \theta_{\langle q \rangle, \langle q^{-1} \rangle} \otimes \theta_{\langle q_1 \rangle, \dots, \langle q_n \rangle}$, which follows from the gluing law of θ (see [55, Section 5.1]).
- (2) $\theta_{\langle q \rangle, \langle q^{-1} \rangle}(e_{\langle q \rangle}, e_{\langle q^{-1} \rangle}) = 1$, which follows from a direct calculation (see [59, Example 6.4]).

The *cutting tree* property (6.9) is proved by a similar argument and we omit the details. \square

The Gromov-Witten theory of BH is completely solved by [38]. It is not hard to see that the methods used in [38] can be used to solve the c_i -twisted Gromov-Witten theory of BQ_i . We do not pursue this here. Instead, we now proceed to compare the Gromov-Witten theory of BH with the c_i -twisted Gromov-Witten theory of BQ_i .

For classes $\phi_1, \dots, \phi_n \in H^\bullet(IBH, \mathbb{C})$ and integers $a_1, \dots, a_n \geq 0$, denote by

$$\langle \tau_{a_1}(\phi_1), \dots, \tau_{a_n}(\phi_n) \rangle_{g,n}^{BH}$$

the corresponding descendant Gromov-Witten invariant of BH . Its definition can be found in [38, Section 3]. Let

$$IBH = \bigcup_{\langle h \rangle \subset H} BC_H(h)$$

¹⁵The general construction of [55] implies that gerbe-twisted Gromov-Witten invariants form a cohomological field theory in general.

be the decomposition of the inertia orbifold of BH , where the union is taken over conjugacy classes $\langle h \rangle$ of H , and $C_H(h) \subset H$ is the centralizer subgroup of $h \in H$. For each conjugacy class $\langle h \rangle$ of H , let $1_{\langle h \rangle} := 1 \in H^0(BC_H(h), \mathbb{C})$ be the natural generator. By [38, Proposition 3.4], we have

$$(6.11) \quad \langle \tau_{a_1}(1_{\langle h_1 \rangle}), \dots, \tau_{a_n}(1_{\langle h_n \rangle}) \rangle_{g,n}^{BH} = \langle \tau_{a_1}, \dots, \tau_{a_n} \rangle_{g,n} \Omega_g^H(\langle h_1 \rangle, \dots, \langle h_n \rangle),$$

where the quantity $\Omega_g^H(\langle h_1 \rangle, \dots, \langle h_n \rangle)$ is given by

$$(6.12) \quad \Omega_g^H(\langle h_1 \rangle, \dots, \langle h_n \rangle) = \frac{1}{|H|} \# \left\{ \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \sigma_1, \dots, \sigma_n \mid \prod_{j=1}^g [\alpha_j, \beta_j] = \prod_{k=1}^n \sigma_k, \sigma_k \in \langle h_k \rangle \text{ for all } k \right\}.$$

By [38, Lemma 3.5], the quantity $\Omega_g^H(\langle h_1 \rangle, \dots, \langle h_n \rangle)$ satisfies the following properties:

$$(6.13) \quad \Omega_g^H(1_{\langle h_1 \rangle}, \dots, 1_{\langle h_n \rangle}) = \Omega_g^H(1_{\langle 1 \rangle}, 1_{\langle h_1 \rangle}, \dots, 1_{\langle h_n \rangle}),$$

$$(6.14) \quad \Omega_g^H(1_{\langle h_1 \rangle}, \dots, 1_{\langle h_n \rangle}) = \sum_{\langle h \rangle} |C_H(h)| \Omega_{g-1}^H(1_{\langle h \rangle}, 1_{\langle h^{-1} \rangle}, 1_{\langle h_1 \rangle}, \dots, 1_{\langle h_n \rangle}),$$

and

$$(6.15) \quad \Omega_g^H(1_{\langle h_1 \rangle}, \dots, 1_{\langle h_n \rangle}) = \sum_{\langle h \rangle} |C_H(h)| \Omega_{g_1}^H(\{1_{\langle h_i \rangle}\}_{i \in P_1}, 1_{\langle h \rangle}) \Omega_{g_2}^H(1_{\langle h^{-1} \rangle}, \{1_{\langle h_i \rangle}\}_{i \in P_2}),$$

where $g = g_1 + g_2$ and $\{1, \dots, n\} = P_1 \amalg P_2$.

Theorem 6.5. *Let*

$$\{\phi_{ij}\}_{1 \leq j \leq \dim H_{orb, c_i}^\bullet(BQ_i)} \subset H_{orb, c_i}^\bullet(BQ_i)$$

be an additive basis. We also view ϕ_{ij} as an element in $QH_{orb}^\bullet(BH)$ via (6.3). Then

$$(6.16) \quad \langle \tau_{a_1}(\phi_{i_1 j_1}), \dots, \tau_{a_n}(\phi_{i_n j_n}) \rangle_{g,n}^{BH} = 0$$

unless $i_1 = i_2 = \dots = i_n =: i$, in which case,

$$(6.17) \quad \langle \tau_{a_1}(\phi_{i j_1}), \dots, \tau_{i j_n}(\phi_{i j_n}) \rangle_{g,n}^{BH} = \left(\frac{\dim V_{\rho_i}}{|G|} \right)^{2-2g} \langle \tau_{a_1}(\phi_{i j_1}), \dots, \tau_{i j_n}(\phi_{i j_n}) \rangle_{g,n}^{BQ_i, c_i}.$$

Proof. We apply the argument in the proof of [38, Proposition 4.2]. In the case of 3-pointed genus 0 invariants, (6.16)-(6.17) follow easily from the isomorphism (6.3) and Lemma 6.2. Then we proceed by induction on the genus g and the number of insertions n . Using (6.5) and (6.11), we can rephrase (6.16)-(6.17) in terms of the quantities $\Lambda_{g,n}^{BQ_i, c_i}(-)$ and $\Omega_g^H(-)$. The induction step can then be carried out using properties (6.7)-(6.9) of $\Lambda_{g,n}^{BQ_i, c_i}(-)$ and properties (6.13)-(6.15) of $\Omega_g^H(-)$. The details are left to the reader. \square

We now use Theorem 6.5 to prove Conjecture 1.9 for the gerbe $BH \rightarrow BQ$. Let

$$\{t_{ij,a} \mid O_i \in \text{Orb}^Q(\widehat{G}), 1 \leq j \leq \dim H_{orb, c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}\}$$

be a set of variables. The genus g descendant Gromov-Witten potentials are defined as follows:

$$\begin{aligned} & \mathcal{F}_{BQ_i, c_i}^g(\{t_{ij,a}\}_{1 \leq j \leq \dim H_{orb, c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}}) \\ &:= \sum_{n \geq 0} \sum_{\substack{a_1, \dots, a_n \geq 0 \\ j_1, \dots, j_n}} \frac{\prod_{k=1}^n t_{ij_k}}{n!} \langle \tau_{a_1}(\phi_{i j_1}), \dots, \tau_{a_n}(\phi_{i j_n}) \rangle_{g,n}^{BQ_i, c_i}, \end{aligned}$$

$$\begin{aligned} & \mathcal{F}_{BH}^g(\{t_{ij,a}\}_{O_i \in \text{Orb}^Q(\widehat{G}), 1 \leq j \leq \dim H_{orb,c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}}) \\ &:= \sum_{n \geq 0} \sum_{\substack{a_1, \dots, a_n \geq 0 \\ i_1, \dots, i_n; j_1, \dots, j_n}} \frac{\prod_{k=1}^n t_{i_k j_k}}{n!} \langle \tau_{a_1}(\phi_{i_1 j_1}), \dots, \tau_{a_n}(\phi_{i_n j_n}) \rangle_{g,n}^{BH}. \end{aligned}$$

The total descendant potentials are defined as follows:

$$\begin{aligned} \mathcal{D}_{BQ_i, c_i}(\{t_{ij,a}\}_{1 \leq j \leq \dim H_{orb,c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}}; \epsilon) &:= \exp \left(\sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_{BQ_i, c_i}^g \right), \\ \mathcal{D}_{BH}(\{t_{ij,a}\}_{O_i \in \text{Orb}^Q(\widehat{G}), 1 \leq j \leq \dim H_{orb,c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}}; \epsilon) &:= \exp \left(\sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_{BH}^g \right). \end{aligned}$$

By Theorem 6.5, we have

$$\begin{aligned} & \mathcal{D}_{BH}(\{t_{ij,a}\}_{O_i \in \text{Orb}^Q(\widehat{G}), 1 \leq j \leq \dim H_{orb,c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}}; \epsilon) \\ &= \sum_{O_i \in \text{Orb}^Q(\widehat{G})} \mathcal{D}_{BQ_i, c_i}(\{t_{ij,a}\}_{1 \leq j \leq \dim H_{orb,c_i}^\bullet(BQ_i), a \in \mathbb{N}_{\geq 0}}; \epsilon \frac{|G|}{\dim V_{\rho_i}}), \end{aligned}$$

which is Conjecture 1.9 in this case.

7. SHEAVES ON GERBES AND TWISTED SHEAVES

In this section, we discuss sheaf theoretic aspects of the duality of gerbes. We will work with sheaves of abelian groups. Our main result states that the category of sheaves on a G -gerbe \mathcal{Y} over \mathcal{B} is equivalent to the category of c -twisted sheaves on its dual $\widehat{\mathcal{Y}}$.

To illustrate our approach to sheaf theory on G -gerbes, we begin with considering (complex) vector bundles on the simplest example of G -gerbes, namely, $BG \rightarrow pt$. In this case, the dual space is \widehat{G} and the $U(1)$ -gerbe c on \widehat{G} is trivial. We aim at relating vector bundles on BG with vector bundles on \widehat{G} .

By definition, a (complex) vector bundle on BG is a \mathbb{C} -linear representation V of G . There is a decomposition of V into a direct sum of irreducible G -representations, i.e.,

$$(7.1) \quad V = \bigoplus_{[\rho] \in \widehat{G}} \text{Hom}_G(V_\rho, V) \otimes V_\rho,$$

where $\text{Hom}_G(V_\rho, V)$ is the \mathbb{C} -vector space of G -equivariant linear maps from V_ρ to V . The collection $\{\text{Hom}_G(V_\rho, V) \mid [\rho] \in \widehat{G}\}$ can be viewed as a vector bundle over the disconnected space \widehat{G} by assigning the vector space $\text{Hom}_G(V_\rho, V)$ to the point $[\rho] \in \widehat{G}$. Thus, the assignment $V \mapsto \{\text{Hom}_G(V_\rho, V) \mid [\rho] \in \widehat{G}\}$ defines a functor from the category of complex vector bundles on BG to the category of complex vector bundles on \widehat{G} . It is easy to see that this functor is an equivalence of abelian categories.

It is clear from the above discussion that the key in the construction of the equivalence $V \mapsto \{\text{Hom}_G(V_\rho, V) \mid [\rho] \in \widehat{G}\}$ is the decomposition (7.1) of a G -representation into a direct sum of irreducible representations. Such a direct sum decomposition will also be the key to our study of the sheaf theory on an arbitrary G -gerbe.

7.1. Global quotient. In this subsection, we study the sheaf theory for gerbes arising from global quotients. Our approach is based on [31, Chapter XII], which can be understood as the case of the G -gerbe $BH \rightarrow BQ$ arising from the extension (3.1).

As in Section 4.2, consider the exact sequence (3.1)

$$1 \rightarrow G \xrightarrow{i} H \xrightarrow{j} Q \rightarrow 1.$$

As in Section 3.1, we choose a section $s : Q \rightarrow H$. Recall that with such a section we can define a cocycle $\tau : Q \times Q \rightarrow G$ via

$$(7.2) \quad s(q_1)s(q_2) = \tau(q_1, q_2)s(q_1q_2), \quad q_1, q_2 \in Q.$$

Let M be a smooth manifold¹⁶ with an H -action. We assume that this H -action restricts to a trivial G -action on M . Consequently, a Q -action on M is naturally defined. The natural map between the quotients

$$(7.3) \quad [M/H] \rightarrow [M/Q]$$

is a G -gerbe. Note, also, that the H and Q actions on M agree in the following sense: let $q \in Q$ and any $h \in j^{-1}(q) \subset H$, then for any point $m \in M$, we have $q(m) = h(m)$. This holds because $G = \ker(j)$ acts trivially on M .

Given a G -representation ρ and $h \in H$ we may consider another G -representation,

$$(7.4) \quad G \ni g \mapsto \rho(hgh^{-1}).$$

It is easy to see that (7.4) defines a right action of H on the set \widehat{G} of *isomorphism classes* of irreducible representations of G . If $h \in G$, then (7.4) is equivalent to ρ via the intertwining operator $\rho(h)$. Hence, by choosing a section $s : Q \rightarrow H$, one sees that (7.4) defines a right action¹⁷ of Q on \widehat{G} as well. This is the action¹⁸ we have seen in Section 3.1. Let H and Q act on the product $M \times \widehat{G}$ by the given actions on the factors. Again, the two actions agree in the following sense: let $q \in Q$ and any $h \in j^{-1}(q) \subset H$, then for any $m \in M$ and $[\rho] \in \widehat{G}$, we have $q((m, [\rho])) = h((m, [\rho]))$. This holds because $G = \ker(j)$ acts trivially on $M \times \widehat{G}$.

As in Section 3.1, for each isomorphism class $[\rho] \in \widehat{G}$, we fix a choice of a representative, denoted by $\rho : G \rightarrow \text{End}(V_\rho)$.

7.1.1. A twisted sheaf. Let \mathcal{V}_G be the sheaf over $M \times \widehat{G}$ defined by requiring that its restriction to $M \times [\rho]$ is the trivial sheaf with fiber V_ρ . There is a natural G -sheaf structure on \mathcal{V}_G . We will construct twisted actions by H and Q , so that \mathcal{V}_G descends to twisted sheaves on the orbifolds $[M \times \widehat{G}/H]$ and $[M \times \widehat{G}/Q]$. We refer to [20] for a discussion on the category of twisted sheaves.

Let $T_q^{[\rho]} : V_\rho \rightarrow V_{q([\rho])}$ be the intertwining operator introduced in Section 3.1. Recall that the following equation holds:

$$(7.5) \quad \rho(s(q)gs(q)^{-1}) = T_q^{[\rho]-1} \circ q([\rho])(g) \circ T_q^{[\rho]}.$$

For $h \in H$, there exists unique $q \in Q$ and $g \in G$ such that $h = gs(q)$. Define

$$E_{h, [\rho]} := \rho(g) \circ T_q^{[\rho]-1} : V_{q([\rho])} \rightarrow V_\rho.$$

For $h \in H$ we may view $E_{h, [\rho]}$ as an isomorphism $h^*\mathcal{V}_G|_{M \times [\rho]} \rightarrow \mathcal{V}_G|_{M \times [\rho]}$, where $h : M \times \widehat{G} \rightarrow M \times \widehat{G}$ is the map defined by the action of $h \in H$.

Similarly, for $q \in Q$, we define $E_{q, [\rho]} := E_{s(q), [\rho]}$ and view it as an isomorphism $q^*\mathcal{V}_G|_{M \times [\rho]} \rightarrow \mathcal{V}_G|_{M \times [\rho]}$, where $q : M \times \widehat{G} \rightarrow M \times \widehat{G}$ is the map defined by the action by $q \in Q$.

¹⁶Depending on the context, we will work with Euclidean, analytic, or étale topology. Our arguments work in all these settings.

¹⁷Note that this Q -action is independent of the choice of the section.

¹⁸Again, we write this action as a left action.

Recall that in Section 3.1, a cocycle $c : \widehat{G} \times Q \times Q \rightarrow U(1)$ is defined. We extend this to a cocycle $c : \widehat{G} \times H \times H \rightarrow U(1)$ as follows. For $h_1, h_2 \in H$, we may uniquely write $h_1 = g_1 s(q_1)$ and $h_2 = g_2 s(q_2)$. Set $c^{[\rho]}(h_1, h_2) := c^{[\rho]}(q_1, q_2)$.

Lemma 7.1.

- (1) The collection $\{E_{h, [\rho]} | h \in H, [\rho] \in \widehat{G}\}$ of isomorphisms defines a c^{-1} -twisted H -equivariant structure on the sheaf \mathcal{V}_G .
- (2) The collection $\{E_{q, [\rho]} | q \in Q, [\rho] \in \widehat{G}\}$ of isomorphisms defines a c^{-1} -twisted Q -equivariant structure on the sheaf \mathcal{V}_G .

Proof. For either statement it suffices to check that the isomorphisms are compatible with the group actions. We do this for the H -action. The check for the Q -action is similar.

Let $h_1, h_2 \in H$. We need to show that the composition

$$E_{h_1, [\rho]} \circ E_{h_2, h_1([\rho])} : h_2^* h_1^* \mathcal{V}_G|_{M \times [\rho]} \rightarrow h_1^* \mathcal{V}_G|_{M \times [\rho]} \rightarrow \mathcal{V}_G|_{M \times [\rho]}$$

coincides, up to a twist, with

$$E_{h_1 h_2, [\rho]} : (h_1 h_2)^* \mathcal{V}_G|_{M \times [\rho]} \rightarrow \mathcal{V}_G|_{M \times [\rho]}.$$

Write $h_1 = g_1 s(q_1)$ and $h_2 = g_2 s(q_2)$ with $g_1, g_2 \in G$ and $q_1, q_2 \in Q$ as above. We compute

$$\begin{aligned} h_1 h_2 &= g_1 s(q_1) g_2 s(q_2) \\ &= g_1 s(q_1) g_2 s(q_1)^{-1} s(q_1) s(q_2) \\ &= g_1 s(q_1) g_2 s(q_1)^{-1} \tau(q_1, q_2) s(q_1 q_2). \end{aligned}$$

Set $\tilde{g} := g_1 s(q_1) g_2 s(q_1)^{-1} \tau(q_1, q_2)$, so $\tilde{g} \in G$. We compute

$$\begin{aligned} E_{h_1 h_2, [\rho]} &= \rho(\tilde{g}) \circ T_{q_1 q_2}^{[\rho]-1} \\ &= \rho(g_1) \circ \rho(s(q_1) g_2 s(q_1)^{-1}) \circ \rho(\tau(q_1, q_2)) \circ T_{q_1 q_2}^{[\rho]-1} \\ &= \rho(g_1) \circ T_{q_1}^{[\rho]-1} \circ q_1([\rho])(g_2) \circ T_{q_1}^{[\rho]} \circ \rho(\tau(q_1, q_2)) \circ T_{q_1 q_2}^{[\rho]-1} \quad \text{by (7.5)} \\ &= \rho(g_1) \circ T_{q_1}^{[\rho]-1} \circ q_1([\rho])(g_2) \circ T_{q_1}^{[\rho]} \circ T_{q_1}^{[\rho]-1} \circ T_{q_2}^{q_1([\rho])^{-1}} c^{[\rho]}(q_1, q_2) \quad \text{by (3.5)} \\ &= \rho(g_1) \circ T_{q_1}^{[\rho]-1} \circ q_1([\rho])(g_2) \circ T_{q_2}^{q_1([\rho])^{-1}} c^{[\rho]}(q_1, q_2) \\ &= c^{[\rho]}(q_1, q_2) E_{h_1, [\rho]} \circ E_{h_2, h_1([\rho])}, \end{aligned}$$

as desired. □

7.1.2. Equivalence. We consider H -equivariant sheaves $\widetilde{\mathcal{W}}$ on $M \times \widehat{G}$ satisfying the following:

Assumption 7.2. The restriction of $\widetilde{\mathcal{W}}$ to $M \times \{[\rho]\}$ is isomorphic, as G -sheaves, to the tensor product of an ordinary sheaf on M and the trivial G -sheaf V_ρ .

Proposition 7.3. Tensoring with \mathcal{V}_G yields an equivalence between the category of c -twisted Q -equivariant sheaves on $M \times \widehat{G}$ and the category of H -equivariant sheaves on $M \times \widehat{G}$ satisfying Assumption 7.2.

Proof. Let \mathcal{W}' be a c -twisted Q -equivariant sheaf on $M \times \widehat{G}$. Let

$$\Gamma_{q, [\rho]} : q^* \mathcal{W}'|_{M \times [\rho]} \rightarrow \mathcal{W}'|_{M \times [\rho]}, \quad q \in Q, [\rho] \in \widehat{G}$$

be the c -twisted Q -action on \mathcal{W}' . Let $\mathcal{W}' \otimes \mathcal{V}_G$ be the sheaf on $M \times \widehat{G}$ defined by

$$(\mathcal{W}' \otimes \mathcal{V}_G)|_{M \times [\rho]} := \mathcal{W}'|_{M \times [\rho]} \otimes \mathcal{V}_G|_{M \times [\rho]}.$$

For $h \in H$, we write $h = gs(q)$ with $g \in G, q \in Q$. We fix such an expression for each $h \in H$. Set

$$\gamma_{h,[\rho]} := \Gamma_{q,[\rho]} \otimes E_{h,[\rho]} : h^*(\mathcal{W}' \otimes \mathcal{V}_G)|_{M \times [\rho]} \rightarrow \mathcal{W}' \otimes \mathcal{V}_G|_{M \times [\rho]}.$$

For $h_1, h_2 \in H$, we calculate

$$\begin{aligned} \gamma_{h_1,[\rho]} \circ \gamma_{h_2,h_1([\rho])} &= (\Gamma_{q_1,[\rho]} \otimes E_{h_1,[\rho]}) \circ (\Gamma_{q_2,q_1([\rho])} \otimes E_{h_2,h_1([\rho])}) \\ &= (c^{[\rho]}(q_1, q_2)^{-1} \Gamma_{q_1 q_2, [\rho]}) \otimes (c^{[\rho]}(q_1, q_2) E_{h_1 h_2, [\rho]}) \\ &= \Gamma_{q_1 q_2, [\rho]} \otimes E_{h_1 h_2, [\rho]} \\ &= \gamma_{h_1 h_2, [\rho]}. \end{aligned}$$

Therefore, the collection

$$(7.6) \quad \{\gamma_{h,[\rho]}\}, \quad h \in H, [\rho] \in \widehat{G},$$

defines a H -equivariant structure on $\mathcal{W}' \otimes \mathcal{V}_G$.

Let \mathcal{U}' and \mathcal{W}' be two c -twisted Q -equivariant sheaves on $M \times \widehat{G}$ with the equivariant structures given, respectively, by $\{\Gamma_{q,[\rho]}^{\mathcal{U}}\}$ and $\{\Gamma_{q,[\rho]}^{\mathcal{W}}\}$, and let $\mathcal{U}' \otimes \mathcal{V}_G$ and $\mathcal{W}' \otimes \mathcal{V}_G$ be the sheaves with H -equivariant structures defined, respectively, by

$$\gamma_{h,[\rho]}^{\mathcal{U}} = \Gamma_{q,[\rho]}^{\mathcal{U}} \otimes E_{h,[\rho]}, \quad \gamma_{h,[\rho]}^{\mathcal{W}} = \Gamma_{q,[\rho]}^{\mathcal{W}} \otimes E_{h,[\rho]},$$

as in (7.6). By Schur's lemma, we have

$$\text{Hom}_H(\mathcal{U}' \otimes \mathcal{V}_G, \mathcal{W}' \otimes \mathcal{V}_G) = \text{Hom}_{c,Q}(\mathcal{U}', \mathcal{W}').$$

Indeed, for $\phi \in \text{Hom}_H(\mathcal{U}' \otimes \mathcal{V}_G, \mathcal{W}' \otimes \mathcal{V}_G)$, the map ϕ is H -equivariant. Hence, it is also G -equivariant. It follows from Schur's lemma that the restriction $\phi|_{M \times [\rho]} : \mathcal{U}' \otimes \mathcal{V}_G|_{M \times [\rho]} \rightarrow \mathcal{W}' \otimes \mathcal{V}_G|_{M \times [\rho]}$ is of the form $\bar{\phi}_{[\rho]} \otimes id$. The H -equivariance of ϕ reads

$$(\bar{\phi}_{[\rho]} \otimes id) \circ \gamma_{h,[\rho]}^{\mathcal{U}} = \gamma_{h,[\rho]}^{\mathcal{W}} \circ (\bar{\phi}_{q([\rho])} \otimes id).$$

This implies that $\bar{\phi}_{[\rho]} \circ \Gamma_{q,[\rho]}^{\mathcal{U}} = \Gamma_{q,[\rho]}^{\mathcal{W}} \circ \bar{\phi}_{q([\rho])}$. Thus, the collection $\{\bar{\phi}_{[\rho]}|[\rho] \in \widehat{G}\}$ of maps defines a c -twisted Q -equivariant map $\bar{\phi} : \mathcal{U}' \rightarrow \mathcal{W}'$ of sheaves on $M \times \widehat{G}$.

Conversely, given a c -twisted Q -equivariant map $\bar{\phi} : \mathcal{U}' \rightarrow \mathcal{W}'$, the map $\bar{\phi} \otimes id : \mathcal{U}' \otimes \mathcal{V}_G \rightarrow \mathcal{W}' \otimes \mathcal{V}_G$ is equivariant with respect to the H -actions defined in (7.6),

$$\begin{aligned} (\bar{\phi}_{[\rho]} \otimes id) \circ \gamma_{h,[\rho]}^{\mathcal{U}} &= (\bar{\phi}_{[\rho]} \otimes id) \circ (\Gamma_{q,[\rho]}^{\mathcal{U}} \otimes E_{h,[\rho]}) \\ &= (\bar{\phi}_{[\rho]} \circ \Gamma_{q,[\rho]}^{\mathcal{U}}) \otimes E_{h,[\rho]} \\ &= (\Gamma_{q,[\rho]}^{\mathcal{W}} \circ \bar{\phi}_{q([\rho])}) \otimes E_{h,[\rho]} \\ &= (\Gamma_{q,[\rho]}^{\mathcal{W}} \otimes E_{h,[\rho]}) \circ (\bar{\phi}_{q([\rho])} \otimes id) \\ &= \gamma_{h,[\rho]}^{\mathcal{W}} \circ (\bar{\phi}_{q([\rho])} \otimes id), \end{aligned}$$

where we used the Q -equivariance of $\bar{\phi}$ in the middle equality.

We have proved that $(-) \otimes \mathcal{V}_G$ is a fully faithful functor from the category of c -twisted Q -equivariant sheaves on $M \times \widehat{G}$ to the category of H -equivariant sheaves on $M \times \widehat{G}$ satisfying Assumption 7.2. To prove that this functor is an equivalence, we construct the inverse functor. Let \mathcal{W} be a H -equivariant sheaf on $M \times \widehat{G}$ satisfying Assumption 7.2. By Assumption 7.2, we can write

$$(7.7) \quad \mathcal{W}|_{M \times [\rho]} = \text{Hom}_G(V_\rho, \mathcal{W}) \otimes V_\rho,$$

as G -sheaves. Let $\widehat{\mathcal{W}}$ be the sheaf over $M \times \widehat{G}$ defined by

$$\widehat{\mathcal{W}}|_{M \times [\rho]} := \text{Hom}_G(V_\rho, \mathcal{W}).$$

We will show that $\widehat{\mathcal{W}}$ is naturally a c -twisted Q -equivariant sheaf.

Let $\gamma_{h,[\rho]} : h^*\mathcal{W}|_{M \times [\rho]} \rightarrow \mathcal{W}|_{M \times [\rho]}$ denote the H -equivariant structure on \mathcal{W} . In view of (7.7), we may assume that

$$(7.8) \quad \gamma_{g,[\rho]} = id \otimes \rho(g) \quad \text{for } g \in G.$$

Note that $h^*\mathcal{W}|_{M \times [\rho]} = \mathcal{H}om_G(V_{h([\rho])}, \mathcal{W}) \otimes V_{h([\rho])}$. Also, note that if we write $h = gs(q)$ with $g \in G$ and $q \in Q$, then $q([\rho]) = h([\rho])$. Precomposing with $T_q^{[\rho]} : V_\rho \rightarrow V_{q([\rho])} = V_{h([\rho])}$ defines an isomorphism $\mathcal{H}om_G(V_{h([\rho])}, \mathcal{W}) \rightarrow \mathcal{H}om_G(V_\rho, \mathcal{W})$, which we denote by $T_{q,[\rho]}^\vee$. Consider the composition

$$\begin{aligned} \mathcal{H}om_G(V_{h([\rho])}, \mathcal{W}) \otimes V_{h([\rho])} &\xrightarrow{\gamma_{h,[\rho]}} \mathcal{W}|_{M \times [\rho]} = \mathcal{H}om_G(V_\rho, \mathcal{W}) \otimes V_\rho \\ &\xrightarrow{T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}} \mathcal{H}om_G(V_{h([\rho])}, \mathcal{W}) \otimes V_{h([\rho])}. \end{aligned}$$

Claim . The map $(T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ \gamma_{h,[\rho]}$ commutes with $id \otimes h([\rho])(g')$ for any $g' \in G$.

Proof of Claim. We first compute

$$\begin{aligned} &(T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ \gamma_{h,[\rho]} \circ (id \otimes h([\rho])(g')) \\ &= (T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ \gamma_{h,[\rho]} \circ \gamma_{g',h([\rho])} \quad \text{by (7.8)} \\ &= (T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ \gamma_{hg',[\rho]} \quad \text{since } \gamma_{h,[\rho]} \text{ is the equivariant structure on } \mathcal{W} \\ &= (T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ \gamma_{hg'h^{-1},[\rho]} \circ \gamma_{h,(hg'h^{-1})([\rho])} \quad \text{by the same reason} \\ &= (T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ (id \otimes \rho(hg'h^{-1})) \circ \gamma_{h,(hg'h^{-1})([\rho])} \quad G \text{ is normal in } H, \text{ so } hg'h^{-1} \in G \\ &= (T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ (id \otimes \rho(hg'h^{-1})) \circ \gamma_{h,[\rho]} \quad \text{since } G \text{ fixes } [\rho] \\ &= (T_{q,[\rho]}^{\vee-1} \otimes (E_{h,[\rho]}^{-1} \circ \rho(hg'h^{-1}))) \circ \gamma_{h,[\rho]}. \end{aligned}$$

Next we compute $E_{h,[\rho]}^{-1} \circ \rho(hg'h^{-1})$. By definition, $E_{h,[\rho]} = \rho(g)T_q^{[\rho]-1}$, where $h = gs(q)$ with $g \in G, q \in Q$. We also have

$$\rho(hg'h^{-1}) = \rho(gs(q)g's(q)^{-1}g^{-1}) = \rho(g)\rho(s(q)g's(q)^{-1})\rho(g)^{-1},$$

since $s(q)gs(q)^{-1} \in G$ because G is normal in H . Now we have

$$\begin{aligned} E_{h,[\rho]}^{-1} \circ \rho(hg'h^{-1}) &= T_q^{[\rho]}\rho(g)^{-1}\rho(g)\rho(s(q)g's(q)^{-1})\rho(g)^{-1} \\ &= T_q^{[\rho]}\rho(s(q)g's(q)^{-1})\rho(g)^{-1} \\ &= T_q^{[\rho]}T_q^{[\rho]-1} \circ q([\rho])(g') \circ T_q^{[\rho]} \circ \rho(g)^{-1} \quad \text{by (7.5)} \\ &= q([\rho])(g') \circ T_q^{[\rho]} \circ \rho(g)^{-1} \\ &= h([\rho])(g') \circ E_{h,[\rho]}^{-1} \quad \text{because } h([\rho]) = q([\rho]). \end{aligned}$$

Combining the above calculations, the claim follows. \square

By the claim and Schur's lemma, we have

$$(T_{q,[\rho]}^{\vee-1} \otimes E_{h,[\rho]}^{-1}) \circ \gamma_{h,[\rho]} = \Gamma'_{h,[\rho]} \otimes id$$

for some sheaf map $\Gamma'_{h,[\rho]} : \mathcal{H}om_G(V_{h([\rho])}, \mathcal{W}) \rightarrow \mathcal{H}om_G(V_{h([\rho])}, \mathcal{W})$. Let

$$\Gamma_{h,[\rho]} := T_{q,[\rho]}^\vee \circ \Gamma'_{h,[\rho]} : h^*\widehat{\mathcal{W}}|_{M \times [\rho]} = \mathcal{H}om_G(V_{h([\rho])}, \mathcal{W}) \rightarrow \mathcal{H}om_G(V_\rho, \mathcal{W}) = \widehat{\mathcal{W}}|_{M \times [\rho]}.$$

Then $\gamma_{h,[\rho]} = \Gamma_{h,[\rho]} \otimes E_{h,[\rho]}$. We check that the collection $\{\Gamma_{h,[\rho]}\}$ defines c -twisted H -equivariant and Q -equivariant structures on $\widehat{\mathcal{W}}$.

Using the properties of $\gamma_{h,[\rho]}$ and $E_{h,[\rho]}$ that were discussed above, we compute

$$\begin{aligned}
 \Gamma_{h_1 h_2, [\rho]} \otimes id &= (id \otimes E_{h_1 h_2, [\rho]}^{-1}) \circ \gamma_{h_1 h_2, [\rho]} \\
 &= c^{[\rho]}(h_1, h_2)^{-1} (id \otimes E_{h_2, h_1([\rho])}^{-1} \circ E_{h_1, [\rho]}^{-1}) \circ \gamma_{h_1, [\rho]} \circ \gamma_{h_2, h_1([\rho])} \\
 &= c^{[\rho]}(h_1, h_2)^{-1} (id \otimes E_{h_2, h_1([\rho])}^{-1}) \circ (id \otimes E_{h_1, [\rho]}^{-1}) \circ \gamma_{h_1, [\rho]} \circ \gamma_{h_2, h_1([\rho])} \\
 &= c^{[\rho]}(h_1, h_2)^{-1} (id \otimes E_{h_2, h_1([\rho])}^{-1}) \circ (\Gamma_{h_1, [\rho]} \otimes id) \circ \gamma_{h_2, h_1([\rho])} \\
 &= c^{[\rho]}(h_1, h_2)^{-1} (\Gamma_{h_1, [\rho]} \otimes id) \circ (id \otimes E_{h_2, h_1([\rho])}^{-1}) \circ \gamma_{h_2, h_1([\rho])} \\
 &= c^{[\rho]}(h_1, h_2)^{-1} (\Gamma_{h_1, [\rho]} \otimes id) \circ (\Gamma_{h_2, h_1([\rho])} \otimes id) \\
 &= c^{[\rho]}(h_1, h_2)^{-1} (\Gamma_{h_1, [\rho]} \circ \Gamma_{h_2, h_1([\rho])} \otimes id).
 \end{aligned}$$

Thus,

$$(7.9) \quad \Gamma_{h_1 h_2, [\rho]} = c^{[\rho]}(h_1, h_2)^{-1} \Gamma_{h_1, [\rho]} \circ \Gamma_{h_2, h_1([\rho])}.$$

In other words, $\{\Gamma_{h,[\rho]} | h \in H, [\rho] \in \widehat{G}\}$ defines a c -twisted H -equivariant structure on $\widehat{\mathcal{W}}$.

Note that $\Gamma_{g,[\rho]} = id$ for $g \in G$, by (7.8). A special case of (7.9) reads

$$\Gamma_{g,[\rho]} \circ \Gamma_{h,[\rho]} = c^{[\rho]}(g, h) \Gamma_{gh, [\rho]},$$

where $g \in G, h \in H$ and note that $g([\rho]) = [\rho]$. We claim that $c^{[\rho]}(g, h) = 1$ for all $g \in G, h \in H$. To see this, note that by Proposition 3.1, $c^{[\rho]}(g, h) = c^{[\rho]}(1, q) = 1$ for $q \in Q$ such that $hs(q)^{-1} \in G$.

By the discussion above, we find that $\Gamma_{gh, [\rho]} = \Gamma_{h, [\rho]}$ for all $g \in G$. Therefore, $\Gamma_{h, [\rho]}$ depends only on the G -coset of h , not the element h itself. Moreover, for $q \in Q$, the definition

$$\Gamma_{q, [\rho]} := \Gamma_{s(q), [\rho]}$$

is independent of the choice of the section $s : Q \rightarrow H$. It follows from (7.9) that $\{\Gamma_{q, [\rho]} | q \in Q, [\rho] \in \widehat{G}\}$ defines a c -twisted Q -equivariant structure on $\widehat{\mathcal{W}}$.

It is straightforward to check that the functor $\mathcal{W} \mapsto \widehat{\mathcal{W}}$ is the inverse of the functor $\mathcal{W}' \mapsto \mathcal{W}' \otimes \mathcal{V}_G$. The proposition is proved. \square

Lemma 7.4. *The category of H -equivariant sheaves on $M \times \widehat{G}$ satisfying Assumption 7.2 is equivalent to the category of H -equivariant sheaves on M .*

Proof. Let $\widetilde{\mathcal{W}}$ be a H -equivariant sheaf on $M \times \widehat{G}$. Then the direct sum $\bigoplus_{[\rho] \in \widehat{G}} \widetilde{\mathcal{W}}|_{M \times \{[\rho]\}}$ is a sheaf on M with a natural H -equivariant structure induced from that of $\widetilde{\mathcal{W}}$. Clearly,

$$Hom_H(\widetilde{\mathcal{U}}, \widetilde{\mathcal{W}}) = Hom_H\left(\bigoplus_{[\rho] \in \widehat{G}} \widetilde{\mathcal{U}}|_{M \times \{[\rho]\}}, \bigoplus_{[\rho] \in \widehat{G}} \widetilde{\mathcal{W}}|_{M \times \{[\rho]\}}\right).$$

Hence, the assignment $\widetilde{\mathcal{W}} \mapsto \bigoplus_{[\rho] \in \widehat{G}} \widetilde{\mathcal{W}}|_{M \times \{[\rho]\}}$ is a covariant fully faithful functor from the category of H -equivariant sheaves on $M \times \widehat{G}$ satisfying Assumption 7.2 to the category of H -equivariant sheaves on M . It remains to construct an inverse functor.

Let \mathcal{W} be a H -equivariant sheaf on M . Since G acts trivially on M , we have the following canonical decomposition as G -equivariant sheaves (see, e.g., [17], Section 4.2):

$$(7.10) \quad \mathcal{W} = \bigoplus_{[\rho] \in \widehat{G}} Hom_G(V_\rho, \mathcal{W}) \otimes V_\rho,$$

where V_ρ is again the trivial vector bundle over M with a G -action given by ρ , and $\mathcal{H}om_G(V_\rho, \mathcal{W})$ is just an ordinary sheaf on M . Define a sheaf $\widetilde{\mathcal{W}}$ on $M \times \widehat{G}$ by

$$\widetilde{\mathcal{W}}|_{M \times \{[\rho]\}} := \mathcal{H}om_G(V_\rho, \mathcal{W}) \otimes V_\rho.$$

Clearly, $\widetilde{\mathcal{W}}$ satisfies Assumption 7.2. We claim that $\widetilde{\mathcal{W}}$ has the structure of a H -equivariant sheaf.

Since the projection $\mathbf{p} : M \times \widehat{G} \rightarrow M$ onto the first factor is H -equivariant, the pull-back $\mathbf{p}^*\mathcal{W}$ is a H -equivariant sheaf on $M \times \widehat{G}$. Clearly, for any $[\rho] \in \widehat{G}$, we have $\mathbf{p}^*\mathcal{W}|_{M \times \{[\rho]\}} = \mathcal{W}$. Also,

$$\mathcal{H}om_G(\mathcal{V}_G, \mathbf{p}^*\mathcal{W})|_{M \times \{[\rho]\}} = \mathcal{H}om_G(V_\rho, \mathcal{W}).$$

It follows that $\mathcal{H}om_G(V_\rho, \mathcal{W}) \otimes V_\rho = \mathcal{H}om_G(\mathcal{V}_G, \mathbf{p}^*\mathcal{W})|_{M \times \{[\rho]\}} \otimes \mathcal{V}_G|_{M \times \{[\rho]\}}$, i.e.,

$$\widetilde{\mathcal{W}} = \mathcal{H}om_G(\mathcal{V}_G, \mathbf{p}^*\mathcal{W}) \otimes \mathcal{V}_G.$$

By Lemma 7.1, \mathcal{V}_G is a c^{-1} -twisted H -equivariant sheaf. Hence, $\mathcal{H}om_G(\mathcal{V}_G, \mathbf{p}^*\mathcal{W})$ is a c -twisted H -equivariant sheaf, and $\widetilde{\mathcal{W}}$, being the tensor product of $\mathcal{H}om_G(\mathcal{V}_G, \mathbf{p}^*\mathcal{W})$ and \mathcal{V}_G , is a H -equivariant sheaf.

For two H -equivariant sheaves \mathcal{U} and \mathcal{W} on M , it is easy to see that $\mathcal{H}om_H(\mathcal{U}, \mathcal{W}) = \mathcal{H}om_H(\widetilde{\mathcal{U}}, \widetilde{\mathcal{W}})$. The functor $\mathcal{W} \mapsto \widetilde{\mathcal{W}}$ provides the needed inverse functor. \square

It is known that sheaves on the orbifold $[M/H]$ are equivalent to H -equivariant sheaves on M . The cocycle c defines a cocycle, which we still denote by c , on the underlying orbifold $[M \times \widehat{G}/Q]$. It is also known that c -twisted sheaves on the orbifold $[M \times \widehat{G}/Q]$ are equivalent to c -twisted Q -equivariant sheaves on $M \times \widehat{G}$. Thus, we may combine Proposition 7.3 with Lemma 7.4 and rephrase the above result as the following:

Theorem 7.5. *The category of sheaves on the gerbe $[M/H]$ is equivalent to the category of c -twisted sheaves on $[M \times \widehat{G}/Q]$.*

7.2. General case. In this section we discuss sheaf theory on a general G -gerbe. Let $\mathcal{Y} \rightarrow \mathcal{B}$ be a G -gerbe over an orbifold \mathcal{B} . If $\mathfrak{H} \rightrightarrows \mathfrak{H}_0$ is an étale groupoid presenting the gerbe \mathcal{Y} , then sheaves on \mathcal{Y} are equivalent to \mathfrak{H} -sheaves on \mathfrak{H}_0 . Therefore, in order to study sheaves on the gerbe \mathcal{Y} , we may pick a suitable étale groupoid presentation \mathfrak{H} of \mathcal{Y} and work with \mathfrak{H} -sheaves.

As discussed in Section 4.2, we may choose proper étale groupoids \mathfrak{H} and \mathfrak{Q} so that the G -gerbe $\mathcal{Y} \rightarrow \mathcal{B}$ is presented by the groupoid extension

$$(7.11) \quad (M \times G \rightrightarrows M) \xrightarrow{i} \mathfrak{H} \xrightarrow{j} \mathfrak{Q},$$

such that

- (1) $M \times G \rightrightarrows M$ is the groupoid for the trivial action of G on M .
- (2) $\mathfrak{H} \rightrightarrows \mathfrak{H}_0$ with $\mathfrak{H}_0 = M$ is a presentation of \mathcal{Y} .
- (3) $\mathfrak{Q} \rightrightarrows \mathfrak{Q}_0$ with $\mathfrak{Q}_0 = M$ is a presentation of \mathcal{B} .
- (4) $i|_M = j|_M = \text{identity}$.
- (5) there is a section of j , i.e., a map $\sigma : \mathfrak{Q} \rightarrow \mathfrak{H}$, such that $j \circ \sigma = \text{id}$ and $\sigma|_M = \text{identity}$.

Remark 7.6. *In the algebraic context, namely, when \mathcal{Y} and \mathcal{B} are Deligne-Mumford stacks over \mathbb{C} , a presentation of $\mathcal{Y} \rightarrow \mathcal{B}$ as in (7.11) can be obtained as follows. Using [4], Lemma 2.2.3, we may find an étale cover $U := \coprod_i U_i \rightarrow \mathcal{Y}$ such that \mathcal{Y} is locally isomorphic to a quotient $[U_i/H_i]$ for some finite group H_i acting on U_i . Since \mathcal{Y} is a G -gerbe, the group H_i contains G as a normal subgroup, and the induced G -action on U_i is trivial. Set $Q_i := H_i/G$. Then \mathcal{B} is locally isomorphic to the quotient $[U_i/Q_i]$, and the map $\mathcal{Y} \rightarrow \mathcal{B}$ is locally presented as $[U_i/H_i] \rightarrow [U_i/Q_i]$. We may take $M = U$, $\mathfrak{H} := (U \times_{\mathcal{Y}} U \rightrightarrows U)$, and $\mathfrak{Q} := (U \times_{\mathcal{B}} U \rightrightarrows U)$. Choosing a section $\sigma : \mathfrak{Q} \rightarrow \mathfrak{H}$ amounts to choosing*

sections $Q_i \rightarrow H_i$ and $Q_{ij} \rightarrow H_{ij}$, where Q_{ij} and H_{ij} are finite groups, so that over $U_i \times_{\mathcal{Y}} U_j$, the map $\mathcal{Y} \rightarrow \mathcal{B}$ is presented as $[V_{ij}/H_{ij}] \rightarrow [V_{ij}/Q_{ij}]$.

We now proceed to study sheaves on the gerbe \mathcal{Y} by studying \mathfrak{H} -sheaves on $M = \mathfrak{H}_0$, in a way similar to the treatment in Section 7.1.

As discussed in Section 4.2, the groupoid Ω acts on \widehat{G} . Similarly, \mathfrak{H} acts on \widehat{G} as well. Indeed, the \mathfrak{H} -action on \widehat{G} is obtained from the Ω -action by the map $j : \mathfrak{H} \rightarrow \Omega$. Consider the two transformation groupoids $\widehat{\mathfrak{H}} := \widehat{G} \rtimes \mathfrak{H}$ and $\widehat{\Omega} := \widehat{G} \rtimes \Omega$. There is a groupoid cocycle c on $\widehat{\Omega}$. Note that $\widehat{\mathfrak{H}}_0 = \widehat{\Omega}_0 = M \times \widehat{G}$.

Let \mathcal{V}_G be the sheaf over $M \times \widehat{G}$ defined by requiring that its restriction to $M \times [\rho]$ be the trivial sheaf with fiber V_ρ . There is a natural G -sheaf structure on \mathcal{V}_G . Similar to the method in Section 7.1, we can construct a c -twisted $\widehat{\mathfrak{H}}$ -equivariant (respectively, $\widehat{\Omega}$ -equivariant) structure on \mathcal{V}_G . Here, c is the cocycle in Proposition 4.6. Evidently, c can be extended to a cocycle $\widehat{\mathfrak{H}} \times_{\widehat{\mathfrak{H}}_0} \widehat{\mathfrak{H}} \rightarrow U(1)$ via the map $j : \mathfrak{H} \rightarrow \Omega$.

We consider \mathfrak{H} -sheaves $\widetilde{\mathcal{W}}$ on $M \times \widehat{G}$ satisfying the following:

Assumption 7.7. The restriction of $\widetilde{\mathcal{W}}$ to $M \times \{[\rho]\}$ is isomorphic, as G -sheaves, to the tensor product of an ordinary sheaf on M and the trivial G -sheaf V_ρ .

The generalization of Proposition 7.3 reads as follows:

Proposition 7.8. *Tensoring with \mathcal{V}_G yields an equivalence between the category of c -twisted $\widehat{\Omega}$ -sheaves on $M \times \widehat{G}$ and the category of $\widehat{\mathfrak{H}}$ -sheaves on $M \times \widehat{G}$ satisfying Assumption 7.7.*

Proof. The proof is a straightforward modification of the proof of Proposition 7.3. We omit the details. \square

Let \mathcal{W} be a \mathfrak{H} -sheaf on M . Since G acts trivially on M , we have the following canonical decomposition as G -equivariant sheaves:

$$(7.12) \quad \mathcal{W} = \bigoplus_{[\rho] \in \widehat{G}} \mathcal{H}om_G(V_\rho, \mathcal{W}) \otimes V_\rho,$$

where V_ρ is, again, the trivial vector bundle over M with a G -action given by ρ , and $\mathcal{H}om_G(V_\rho, \mathcal{W})$ is just an ordinary sheaf on M .

Given \mathcal{W} as above, define a sheaf $\widetilde{\mathcal{W}}$ on $M \times \widehat{G}$ by

$$\widetilde{\mathcal{W}}|_{M \times \{[\rho]\}} := \mathcal{H}om_G(V_\rho, \mathcal{W}) \otimes V_\rho.$$

A generalization of Lemma 7.4 is immediate. We omit the proof.

Lemma 7.9. *The assignment $\mathcal{W} \mapsto \widetilde{\mathcal{W}}$ defines an equivalence between the category of \mathfrak{H} -sheaves on M and the category of $\widehat{\mathfrak{H}}$ -sheaves on $M \times \widehat{G}$ satisfying Assumption 7.7.*

Combining Proposition 7.8 with Lemma 7.9, we obtain the following:

Theorem 7.10. *The category of \mathfrak{H} -sheaves on M is equivalent to the category of c -twisted $\widehat{\Omega}$ -sheaves on $M \times \widehat{G}$.*

Observe that the groupoid $\widehat{\Omega}$ is a presentation of the orbifold $\widehat{\mathcal{Y}}$, which is dual to the gerbe $\mathcal{Y} \rightarrow \mathcal{B}$, and the cocycle c defines a flat $U(1)$ -gerbe on $\widehat{\mathcal{Y}}$ (which we still denote by c). It is known that sheaves on \mathcal{Y} are equivalent to \mathfrak{H} -sheaves on M , and c -twisted sheaves on $\widehat{\mathcal{Y}}$ are equivalent to c -twisted Ω -sheaves on $M \times \widehat{G}$. Therefore, we may rephrase the above theorem as follows:

Theorem 7.11. *The category of sheaves on \mathcal{Y} is equivalent to the category of c -twisted sheaves on $\widehat{\mathcal{Y}}$.*

Remark 7.12.

- (1) *It is clear that our arguments in this section are valid in the algebro-geometric context. Hence, the main results of this section, as well as their counterparts for (quasi-)coherent sheaves, hold for G -gerbes over Deligne-Mumford stacks as well. For example, the abelian category of (quasi-)coherent sheaves on \mathcal{Y} is equivalent to the abelian category of c -twisted (quasi-)coherent sheaves on $\widehat{\mathcal{Y}}$.*
- (2) *The equivalences in Theorems 7.11 and 7.5 do not preserve tensor product structure. The main reason is that the tensor product of two c -twisted sheaves is not a c -twisted sheaf. However, the equivalence is compatible with tensor products by “invariant” vector bundles. This means the following. Let $F : \mathrm{Sh}(\mathcal{Y}) \rightarrow \mathrm{Sh}(\widehat{\mathcal{Y}})$ be the equivalence in Theorem 7.11. Let $\pi_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{B}$ and $\pi_{\widehat{\mathcal{Y}}} : \widehat{\mathcal{Y}} \rightarrow \mathcal{B}$ denote the natural maps. Suppose that $V \rightarrow \mathcal{B}$ is a vector bundle such that at any point $x \in \mathcal{B}$, the action of the isotropy group $\mathrm{Iso}(x)$ on the fiber V_x is trivial¹⁹. Then for any sheaf \mathcal{F} on \mathcal{Y} , we have*

$$F(\mathcal{F} \otimes \pi_{\mathcal{Y}}^* V) = F(\mathcal{F}) \otimes \pi_{\widehat{\mathcal{Y}}}^* V.$$

This follows easily from the construction of the functor F .

APPENDIX A. SOME RESULTS ON FINITE GROUP EXTENSIONS

Consider an extension of finite groups as in (3.1)

$$(A.1) \quad 1 \longrightarrow G \xrightarrow{i} H \xrightarrow{j} Q \longrightarrow 1.$$

In this appendix, we discuss some group-theoretic applications of our analysis of the group algebra $\mathbb{C}H$ in Section 3.

A.1. Counting conjugacy classes in group extensions. Let $j : H \rightarrow Q$ be a surjective homomorphism of finite groups. Let $\langle q \rangle \subset Q$ be a conjugacy class of Q . The pre-image

$$j^{-1}(\langle q \rangle) \subset H$$

may be partitioned into a disjoint union of conjugacy classes of H . It is natural to ask the following:

Question A.1. *How many conjugacy classes of H are contained in $j^{-1}(\langle q \rangle)$?*

In this appendix, we discuss an answer to this question.

Let G be the kernel of $j : H \rightarrow Q$. Then we are in the situation of the exact sequence (A.1). The homomorphism $j : H \rightarrow Q$ induces a surjective homomorphism $\mathbf{j} : \mathbb{C}H \rightarrow \mathbb{C}Q$ between group algebras. This, in turn, induces a surjective homomorphism $\mathbf{j} : Z(\mathbb{C}H) \rightarrow Z(\mathbb{C}Q)$ between centers. The centers $Z(\mathbb{C}H)$ and $Z(\mathbb{C}Q)$, viewed as vector spaces, admit natural bases, $\{1_{\langle h \rangle}\} \subset Z(\mathbb{C}H)$ and $\{1_{\langle q \rangle}\} \subset Z(\mathbb{C}Q)$, indexed by conjugacy classes. These bases satisfy the requirement that if $j(\langle h \rangle) = \langle q \rangle$, then $\mathbf{j}(1_{\langle h \rangle}) \in \mathbb{C}1_{\langle q \rangle}$. Let

$$Z(\mathbb{C}H)_{\langle q \rangle} := \bigoplus_{\langle h \rangle \subset j^{-1}(\langle q \rangle)} \mathbb{C}1_{\langle h \rangle}.$$

By construction, the dimension $\dim Z(\mathbb{C}H)_{\langle q \rangle}$ is the number of conjugacy classes of H that are contained in $j^{-1}(\langle q \rangle)$. By Proposition 3.4, the isomorphism $I : Z(\mathbb{C}H) \rightarrow Z(C(\widehat{G} \rtimes Q, c))$ restricts to an additive isomorphism

$$Z(\mathbb{C}H)_{\langle q \rangle} \simeq Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}.$$

We compute the dimension $\dim Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$.

¹⁹If \mathcal{B} is a Deligne-Mumford stack, such a vector bundle V is pulled back from the coarse moduli space of \mathcal{B} .

Let $\widehat{G}^q \subset \widehat{G}$ be the subset consisting of elements fixed by $q \in Q$. Let $C(q) \subset Q$ be the centralizer subgroup of q . Then we have that $Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$ is additively isomorphic to the c -twisted orbifold cohomology $H_{orb}^\bullet([\widehat{G}/C(q)], c)$. Decompose \widehat{G}^q into a disjoint union of $C(q)$ -orbits:

$$(A.2) \quad \widehat{G}^q = \coprod O_i.$$

For each $C(q)$ -orbit O_i , pick a representative $[\rho_i]$ and denote by $Q_i := \text{Stab}_{C(q)}([\rho_i]) \subset C(q)$ the stabilizer subgroup of $[\rho_i]$. Consider the homomorphism

$$\gamma_{-,q}^{[\rho_i]} : C(q) \rightarrow \mathbb{C}^*, \quad C(q) \ni q_1 \mapsto \gamma_{q_1,q}^{[\rho_i]} := c^{[\rho_i]}(q_1, q) c^{[\rho_i]}(q, q_1)^{-1}.$$

Here, $c^{[\rho]}(-, -)$ is the cocycle defined in (3.5). It follows from (A.2) that

$$H_{orb}^\bullet([\widehat{G}^q/C(q)], c) \simeq \bigoplus H_{orb}^\bullet(BQ_i, c).$$

By [59, Example 6.4], we have that $H_{orb}^\bullet(BQ_i, c) = \mathbb{C}$ if the following condition holds:

$$(A.3) \quad \gamma_{q_1,q}^{[\rho_i]} = 1 \quad \text{for all } q_1 \in Q_i.$$

Moreover, if (A.3) does not hold, then $H_{orb}^\bullet(BQ_i, c) = 0$. It follows that $\dim Z(C(\widehat{G} \rtimes Q, c))_{\langle q \rangle}$ is equal to

$$(A.4) \quad \#\{O_i = C(q)\text{-orbit of } \widehat{G}^q \mid \text{there exists } [\rho_i] \in O_i \text{ s.t. } \gamma_{q_1,q}^{[\rho_i]} = 1 \text{ for all } q_1 \in Q_i = \text{Stab}_{C(q)}([\rho_i])\}.$$

If the group G is abelian, then all irreducible representations of G are 1-dimensional, and all intertwiners in (3.5) can be taken to be the identity. In this case, (A.4) can be simplified to

$$(A.5) \quad \#\{O_i = C(q)\text{-orbit of } \widehat{G}^q \mid \text{there exists } [\rho_i] \in O_i, \\ \text{s.t. } \rho_i(\tau(q_1, q)\tau(q, q_1)^{-1}) = 1 \text{ for all } q_1 \in Q_i = \text{Stab}_{C(q)}([\rho_i])\}.$$

If the group G is abelian and H is a semi-direct product of G and Q , then the cocycle $\tau(-, -)$ can be taken to be trivial. In this case, (A.4) can be simplified to

$$(A.6) \quad \#\{C(q)\text{-orbit of } \widehat{G}^q\}.$$

If the Q -action on \widehat{G} is trivial²⁰, then $\widehat{G}^q = \widehat{G}$, and all intertwiners in (3.5) can be taken to be the identity. In this case, (A.4) can be simplified to

$$(A.7) \quad \#\{[\rho] = \text{isomorphism class of irreducible } G\text{-representations} \mid \\ \rho(\tau(q_1, q)\tau(q, q_1)^{-1}) = 1 \text{ for all } q_1 \in C(q)\}.$$

(A.4)–(A.7) provide our answer to Question A.1. To the best of our knowledge, our answer is new. We thank I. M. Isaacs for discussions related to Question A.1.

A.2. An orthogonality relation of characters. The discussion in this appendix is inspired by the proof of the orthogonality relation given in [14, Chapter 2, Section 12]. Consider (A.1) again. The group $H \times H$ acts naturally on the group algebra $\mathbb{C}H$ via

$$(h_1, h_2) \cdot h = h_1^{-1} h h_2.$$

²⁰Equivalently, this means that the band of the gerbe $BH \rightarrow BQ$ is trivial.

In this way, we may view $\mathbb{C}H$ as a representation of $H \times H$. Its character χ can be calculated as follows:

$$\begin{aligned}\chi((h_1, h_2)) &= \#\{h \in H \mid h_1^{-1} h h_2 = h\} \\ &= \#\{h \in H \mid h h_2 h^{-1} = h_1\} \\ &= \begin{cases} |C_H(h_1)|, & \text{if } h_1 \text{ and } h_2 \text{ are conjugate in } H, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

We now consider $\mathbb{C}H$ as a representation of the subgroup $G \times G$. The above calculation gives the character of this representation: for $(g_1, g_2) \in G \times G$,

$$(A.8) \quad \chi((g_1, g_2)) = \begin{cases} |C_H(g_1)|, & \text{if } g_1 \text{ and } g_2 \text{ are conjugate in } H, \\ 0 & \text{otherwise.} \end{cases}$$

We calculate the character χ by another method. By Proposition 3.2, there is an isomorphism of algebras

$$\mathbb{C}H \simeq \bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \rtimes_{T, \tau} Q.$$

Under this isomorphism, the $G \times G$ action on $\mathbb{C}H$ is identified with the following $G \times G$ action on $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \rtimes_{T, \tau} Q$:

$$\begin{aligned}(g_1, g_2) \cdot (x_\rho, q) &:= \left(\sum_{\rho_1} \rho_1(g_1^{-1}), 1 \right) \circ (x_\rho, q) \circ \left(\sum_{\rho_2} \rho_2(g_2), 1 \right) \\ &= (\rho(g_1^{-1}) x_\rho T_q^{[\rho]-1} q([\rho])(g_2) T_q^{[\rho]}, q).\end{aligned}$$

Here, \circ is the algebraic structure on $\bigoplus_{[\rho] \in \widehat{G}} \text{End}(V_\rho) \rtimes_{T, \tau} Q$.

For each ρ , fix an isomorphism of $\text{End}(V_\rho)$ with a matrix algebra, and let e_{st}^ρ denote the standard basis of this matrix algebra. We use the symbol $(x_\rho)_{st}$ to denote the s, t -entry of $x_\rho \in \text{End}(V_\rho)$. Then we have $(\rho(g_1^{-1}) e_{st}^\rho T_q^{[\rho]-1} q([\rho])(g_2) T_q^{[\rho]})_{st} = (\rho(g_1^{-1}))_{ss} (T_q^{[\rho]-1} q([\rho])(g_2) T_q^{[\rho]})_{tt}$. Therefore,

$$\begin{aligned}\text{tr}((g_1, g_2)|_{\text{End}(V_\rho) \times \{q\}}) &= \sum_{s, t} (\rho(g_1^{-1}))_{ss} (T_q^{[\rho]-1} q([\rho])(g_2) T_q^{[\rho]})_{tt} \\ &= \text{tr}(\rho(g_1^{-1})) \text{tr}(T_q^{[\rho]-1} q([\rho])(g_2) T_q^{[\rho]}) \\ &= \chi_\rho(g_1^{-1}) \chi_{q([\rho])}(g_2),\end{aligned}$$

where χ_ρ and $\chi_{q([\rho])}$ denote the characters of the representations ρ and $q([\rho])$. Summing over $[\rho] \in \widehat{G}$ and $q \in Q$, we find that

$$(A.9) \quad \chi((g_1, g_2)) = \sum_{[\rho] \in \widehat{G}} \sum_{q \in Q} \chi_\rho(g_1^{-1}) \chi_{q([\rho])}(g_2).$$

Combining the above with (A.8), we obtain the following:

$$(A.10) \quad \sum_{[\rho] \in \widehat{G}} \sum_{q \in Q} \chi_\rho(g_1^{-1}) \chi_{q([\rho])}(g_2) = \begin{cases} |C_H(g_1)|, & \text{if } g_1 \text{ and } g_2 \text{ are conjugate in } H, \\ 0 & \text{otherwise.} \end{cases}$$

This can be viewed as a generalization of the orthogonality relations of characters of a finite group.

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